ON DISPERSIONLESS COUPLED MODIFIED KP HIERARCHY

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ABSTRACT. We define and study dispersionless coupled modified KP hierarchy, which incorporates two different versions of dispersionless modified KP hierarchies.

1. INTRODUCTION

Recently, the dispersionless limit of integrable hierarchies is under active research (see, e.g. [Kri92, Kri94, TT92, TT91, TT93, TT95]). There are various problems associated with dispersionless KP (dKP) and dispersionless Toda (dToda) hierarchies, such as topological field theory and its connection to string theory, 2D-gravity, matrix models and conformal maps (see, e.g. [Tak95, Tak96, Dub92a, Dub92b, BX95a, BX95b, Bon02, WZ00, KKMW$^+$01, BMR$^+$01, Zab01, MWZ02]). In contrast, dispersionless modified KP (dmKP) hierarchy is less under spot light. We found at least two different versions of dmKP hierarchies, one is considered by Kupershmidt in [Kup90] and later by Chang and Tu in [CT00], the other is defined by Takebe in [Tak02]. It is well known that a solution of dToda hierarchy will give a solution of dKP hierarchy ([TT95]). In [CT00], Chang and Tu proved that under a Miura map, a solution of dKP hierarchy will give rise to a solution to their dmKP hierarchy. In fact, this process can be reversed and we can view the dmKP hierarchy as a transition between a dToda to a dKP hierarchy. One of the problem that is still left open is the existence of tau function for the dmKP hierarchy. In order that a satisfactory tau function exists, we find that it is necessary to introduce an extra flow to the dmKP hierarchy considered by Kupershmidt, Chang and Tu. This is exactly what Takebe did. Takebe's version of dmKP hierarchy is the dKP hierarchy with an extra flow. Hence we incorporate the two versions of dmKP hierarchy. We consider a dmKP hierarchy in Chang and Tu's version, with an extra time parameter, and call it the dispersionless coupled modified KP hierarchy (dcmKP). We develop the theory along the lines of Takasaki and Takebe [TT95] and Takebe [Tak02].

The basic object in our dcmKP hierarchy is a formal power series $\mathcal{L}$ and a polynomial $\mathcal{P}$ in variable k with coefficients functions of time variables $t = (x, s, t_1, t_2, \ldots)$. We define the hierarchy in terms of Lax equations. We introduce the dressing function in Section 2. By means of the dressing function, we define the Orlov function $\mathcal{M}$, which form a canonical pair with

$\mathcal{L}$, namely $\{\mathcal{L}, \mathcal{M}\} = 1$. We define the S function as a primitive of a closed one form. The tau function is defined so that it generates the coefficients of $\mathcal{M}$.

In Section 3, we prove that a solution of the dToda hierarchy give a solution to our dcmKP hierarchy with $\mathcal{P} = k$. In Section 4, we discuss the Miura map which transform a solution of our dcmKP hierarchy to a solution of Takebe's dmKP hierarchy (dKP hierarchy with an extra parameter s). We find our Miura map the inverse of the one considered by Chang and Tu [CT00] when s is considered as a parameter.

In Section 5, we consider the twistor construction of solutions to our dcmKP hierarchy. We also show that every solution of our dcmKP hierarchy has an associated twistor data. In Section 6, we consider the $w_{1+\infty}$ symmetry generated by the action of a Hamiltonian vector field to the twistor data.

2. Dispersionless modified KP hierarchy

2.1. **Lax formalism.** We define dispersionless coupled modified KP hierarchy (dcmKP) by incorporating the definitions of dispersionless modified KP hierarchy (dmKP) of [Kup90, CT00] and [Tak02]. The fundamental quantity

$$\mathcal{L} = k + \sum_{n=0}^{\infty} u_{n+1}(t) k^{-n}$$

is a formal power series in k with coefficients $u_{n+1}(t)$ depend on infinitely many continuous variables $t = (x, s, t_1, t_2, \ldots,)$. We also introduce an auxiliary monic polynomial of degree N

$$\mathcal{P} = k^N + p_{N-1}(t) k^{N-1} + \ldots + p_0(t).$$

The differential equations that govern the deformation of $\mathcal{L}$ with respect to $s, t_1, t_2, \ldots,$ are

(2.1)
$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \qquad \mathcal{B}_n = (\mathcal{L}^n)_{>0},$$

$$\frac{\partial \mathcal{L}}{\partial s} = \{\log \mathcal{P}, \mathcal{L}\},$$

$$\frac{\partial \log \mathcal{P}}{\partial t_n} = \frac{\partial (\mathcal{L}^n)_{\geq 0}}{\partial s} - \{\log \mathcal{P}, \mathcal{B}_n\}.$$

[1]where $(\mathcal{A})_S = (\sum_i A_i k^i)_S = \sum_{i \in S} A_i k^i$ specifies the part of the power series $\mathcal{A}$ to extract, and $\{\cdot, \cdot\}$ is the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial k} \frac{\partial g}{\partial x} - \frac{\partial g}{\partial k} \frac{\partial f}{\partial x}.$$

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[1]We understand that $\log \mathcal{P}$ is formerly $\log k^N + \frac{a_1}{k} + \frac{a_2}{k^2} + \ldots$.

As usual, the t_1 flow says that

$$\frac{\partial \mathcal{L}}{\partial t_1} = \frac{\partial \mathcal{L}}{\partial x}.$$

In other words, the dependence on t_1 and x appear in the combination $t_1 + x$. When there are no dependence on s, the first equation in (2.1) is the dmKP hierarchy defined by [Kup90, CT00]. We introduce an extra parameter s and the second equation in (2.1) determines the dependence of $\mathcal{L}$ on s via the auxiliary polynomial $\mathcal{P}$. The last equation determines the t_n-flow of $\mathcal{P}$. By standard argument, we have the following zero curvature equation

$$\frac{\partial \mathcal{B}_m}{\partial t_n} - \frac{\partial \mathcal{B}_n}{\partial t_m} + \{\mathcal{B}_m, \mathcal{B}_n\} = 0$$

and its dual form

(2.2) $$\frac{\partial (\mathcal{L}^m)_{\leq 0}}{\partial t_n} - \frac{\partial (\mathcal{L}^n)_{\leq 0}}{\partial t_m} - \{(\mathcal{L}^m)_{\leq 0}, (\mathcal{L}^n)_{\leq 0}\} = 0.$$

This gives us the consistency between the t_n flows of $\mathcal{L}$. To prove the consistency between the t_n and s flows of $\mathcal{L}$, we first establish the following.

Proposition 2.1. *There exist a function $\phi(t)$ such that*

(2.3) $$\frac{\partial \phi}{\partial t_n} = (\mathcal{L}^n)_0.$$

Proof. We have to check that we can consistently solve for ϕ. Using the dual form of the zero curvature equation (2.2), we have

$$\frac{\partial (\mathcal{L}^m)_0}{\partial t_n} - \frac{\partial (\mathcal{L}^n)_0}{\partial t_m} - (\{(\mathcal{L}^m)_{\leq 0}, (\mathcal{L}^n)_{\leq 0}\})_0 = 0.$$

However, the last term contains powers ≤ -1 of k. Hence (2.3) is established. ϕ is not unique since we do not specify its dependence on s. $\square$

Now we check that the s and t_n flows of $\mathcal{L}$ are consistent. We have

$$\frac{\partial}{\partial s}\{\mathcal{B}_n, \mathcal{L}\} - \frac{\partial}{\partial t_n}\{\log \mathcal{P}, \mathcal{L}\}$$

$$= \{\frac{\partial \mathcal{B}_n}{\partial s}, \mathcal{L}\} + \{\mathcal{B}_n, \{\log \mathcal{P}, \mathcal{L}\}\} - \{\frac{\partial \log \mathcal{P}}{\partial t_n}, \mathcal{L}\} - \{\log \mathcal{P}, \{\mathcal{B}_n, \mathcal{L}\}\}$$

$$= \{\frac{\partial \mathcal{B}_n}{\partial s} - \frac{\partial \log \mathcal{P}}{\partial t_n} - \{\log \mathcal{P}, \mathcal{B}_n\}, \mathcal{L}\}.$$

Now from the third equation in (2.1), we have

$$\frac{\partial \mathcal{B}_n}{\partial s} - \frac{\partial \log \mathcal{P}}{\partial t_n} - \{\log \mathcal{P}, \mathcal{B}_n\} = -\frac{\partial (\mathcal{L}^n)_0}{\partial s}.$$

This is independent of k. On the other hand hand, from the second equation in (2.1), since the right hand side contains powers ≤ -1 of k, we have

$$\frac{\partial (\mathcal{L})_0}{\partial s} = 0.$$

Together with $\frac{\partial(\mathcal{L}^n)_0}{\partial t_m} = \frac{\partial(\mathcal{L}^m)_0}{\partial t_n}$ (from the proof of Proposition 2.1), we have

$$\frac{\partial(\mathcal{L}^n)_0}{\partial x \partial s} = \frac{\partial(\mathcal{L}^1)_0}{\partial t_n \partial s} = 0.$$

Hence $\frac{\partial(\mathcal{L}^n)_0}{\partial s}$ is independent of x, and the s and t_n flows of $\mathcal{L}$ are consistent.

For the consistency of the t_n flows of $\mathcal{P}$, we use the zero curvature condition, the equation $\frac{\partial(\mathcal{L}^n)_0}{\partial t_m} = \frac{\partial(\mathcal{L}^m)_0}{\partial t_n}$ (from the proof of Proposition 2.1), and the fact $\frac{\partial(\mathcal{L}^n)_0}{\partial s}$ is independent of k and x,

$$\frac{\partial}{\partial t_m}\left(\frac{\partial(\mathcal{L}^n)_{\geq 0}}{\partial s} - \{\log\mathcal{P}, \mathcal{B}_n\}\right) - \frac{\partial}{\partial t_n}\left(\frac{\partial(\mathcal{L}^m)_{\geq 0}}{\partial s} - \{\log\mathcal{P}, \mathcal{B}_m\}\right)$$

$$= \frac{\partial}{\partial s}\left(\frac{\partial(\mathcal{L}^n)_{>0}}{\partial t_m} - \frac{\partial(\mathcal{L}^m)_{>0}}{\partial t_n} + \frac{\partial(\mathcal{L}^n)_0}{\partial t_m} - \frac{\partial(\mathcal{L}^m)_0}{\partial t_n}\right) - \{\log\mathcal{P}, \frac{\partial\mathcal{B}_n}{\partial t_m} - \frac{\partial\mathcal{B}_m}{\partial t_n}\}$$

$$- \{\frac{\partial(\mathcal{L}^m)_{\geq 0}}{\partial s} - \{\log\mathcal{P}, \mathcal{B}_m\}, \mathcal{B}_n\} + \{\frac{\partial(\mathcal{L}^n)_{\geq 0}}{\partial s} - \{\log\mathcal{P}, \mathcal{B}_n\}, \mathcal{B}_m\}$$

$$= \frac{\partial}{\partial s}\left(\frac{\partial\mathcal{B}_n}{\partial t_m} - \frac{\partial\mathcal{B}_m}{\partial t_n} + \{\mathcal{B}_n, \mathcal{B}_m\}\right) - \{\log\mathcal{P}, \frac{\partial\mathcal{B}_n}{\partial t_m} - \frac{\partial\mathcal{B}_m}{\partial t_n} + \{\mathcal{B}_n, \mathcal{B}_m\}\} = 0$$

Hence the t_n flows of $\mathcal{P}$ are consistent.

2.2. **Dressing operator.** As in [TT95], we can establish the existence of a dressing operator $\exp(\mathrm{ad}\,\varphi)$, where $\mathrm{ad}\,f(g) = \{f, g\}$.

Proposition 2.2. *There exists a function* $\varphi = \sum_{n=0}^{\infty} \varphi_n k^{-n}$, *such that*

$$\mathcal{L} = e^{\mathrm{ad}\,\varphi} k,$$

$$\nabla_{t_n,\varphi}\varphi = -(\mathcal{L}^n)_{\leq 0}, \qquad \nabla_{s,\varphi}\varphi = \log\mathcal{P} - \log\mathcal{L}^N,$$

where $\nabla_{t,A} B = \sum_{n=0}^{\infty} \frac{(\mathrm{ad}\,\varphi)^n}{(n+1)!}\frac{\partial A}{\partial t}$. *If* $\tilde{\varphi}$ *is another function satisfying the conditions above, then the function* $\psi = \sum_{n=0}^{\infty} \psi_n k^{-n}$ *defined by* $e^{\mathrm{ad}\,\tilde{\varphi}} = e^{\mathrm{ad}\,\varphi} e^{\mathrm{ad}\,\psi}$ *has constant coefficients, i.e.,* ψ_n's *are independent of* t.

Proof. We sketch the proof here. For details, we refer to Proposition 1.2.1 in [TT95]. Standard argument shows that we can find φ_0 such that

$$\mathcal{L} = e^{\mathrm{ad}\,\varphi_0} k.$$

The first two equations in (2.1) imply that

$$\{e^{-\mathrm{ad}\,\varphi_0}\left(\nabla_{t_n,\varphi_0}\varphi_0 + (\mathcal{L}^n)_{\leq 0}\right), k\} = 0,$$

$$\{e^{-\mathrm{ad}\,\varphi_0}\left(\nabla_{s,\varphi_0}\varphi_0 - \log\mathcal{P} + \log\mathcal{L}^N\right), k\} = 0.$$

Hence $A_n = e^{-\mathrm{ad}\,\varphi_0}\left(\nabla_{t_n,\varphi_0}\varphi_0 + (\mathcal{L}^n)_{\leq 0}\right)$ and $A_0 = e^{-\mathrm{ad}\,\varphi_0}\left(\nabla_{s,\varphi_0}\varphi_0 - \log\mathcal{P} + \log\mathcal{L}^N\right)$ are independent of x. Using Lemma A.3 in the Appendix A of [TT95] and that A_i's are independent of x, we have $\frac{\partial A_n}{\partial t_m} = \frac{\partial A_m}{\partial t_n}$ and $\frac{\partial A_n}{\partial s} = \frac{\partial A_0}{\partial t_n}$. Hence we can find a function $\varphi' = \sum_{n=0}^{\infty} \varphi'_n k^{-n}$ such that

$$\frac{\partial\varphi'}{\partial t_n} = -A_n, \qquad \frac{\partial\varphi'}{\partial s} = -A_0.$$

Moreover, φ' can be chosen so that it is independent of x. Now we define φ by $e^{\operatorname{ad}\varphi} = e^{\operatorname{ad}\varphi_0} e^{\operatorname{ad}\varphi'}$. Since φ' is independent of x, we have $e^{\operatorname{ad}\varphi'} k = k$ and $\nabla_{t_n,\varphi'}\varphi' = \frac{\partial\varphi'}{\partial t_n}$, $\nabla_{s,\varphi'}\varphi' = \frac{\partial\varphi'}{\partial s}$. So we have

$$\mathcal{L} = e^{\operatorname{ad}\varphi} k.$$

Moreover, using Lemma A.2 in Appendix A of [TT95], we have

$$\nabla_{t_n,\varphi}\varphi = \nabla_{t_n,\varphi_0}\varphi_0 + e^{\operatorname{ad}\varphi_0}\nabla_{t_n,\varphi'}\varphi' = -(\mathcal{L}^n)_{\leq 0},$$
$$\nabla_{s,\varphi}\varphi = \nabla_{s,\varphi_0}\varphi_0 + e^{\operatorname{ad}\varphi_0}\nabla_{s,\varphi'}\varphi' = \log\mathcal{P} - \log\mathcal{L}^N.$$

$\square$

2.3. Orlov function and Darboux coordinates.

Using the dressing operator φ, we can construct the Orlov function $\mathcal{M}$ by

$$\mathcal{M} = e^{\operatorname{ad}\varphi}\left(\sum_{n=1}^{\infty} nk^{n-1} + x + \frac{Ns}{k}\right)$$
$$= \sum_{n=1}^{\infty} n\mathcal{L}^{n-1} + x + \frac{Ns}{\mathcal{L}} + \sum_{n=1}^{\infty} v_n\mathcal{L}^{-n-1},$$

where v_n are functions of t. $\mathcal{M}$ has the property that it forms a canonical pair with $\mathcal{L}$, namely $\{\mathcal{L},\mathcal{M}\} = 1$. Using Proposition 2.2 and Lemma A.1 in Appendix A of [TT95], we also find

$$(2.4) \qquad \frac{\partial\mathcal{M}}{\partial t_n} = \{\mathcal{B}_n,\mathcal{M}\}, \qquad \frac{\partial\mathcal{M}}{\partial s} = \{\log\mathcal{P},\mathcal{M}\}.$$

Now we want to express $\mathcal{B}_n$'s and $\log\mathcal{P}$ by v_n's. From definition, we can write the functions $\mathcal{B}_n$ and $\log\mathcal{P}$ as

$$\mathcal{B}_n = \mathcal{L}^n - \frac{\partial\phi}{\partial t_n} - \sum_{m=1}^{\infty} a_{n,m}(t)\mathcal{L}^{-m},$$
$$\log\mathcal{P} = \log\mathcal{L}^N - \sum_{m=1}^{\infty} a_{0,m}(t)\mathcal{L}^{-m},$$

for some functions $a_{n,m}(t)$ of t. They can be expressed in terms of partial derivatives of v_n's with respect to t:

Proposition 2.3.

$$\mathcal{B}_n = \mathcal{L}^n - \frac{\partial\phi}{\partial t_n} - \sum_{m=1}^{\infty} \frac{1}{m}\frac{\partial v_m}{\partial t_n}\mathcal{L}^{-m},$$
$$\log\mathcal{P} = \log\mathcal{L}^N - \sum_{m=1}^{\infty} \frac{1}{m}\frac{\partial v_m}{\partial s}\mathcal{L}^{-m}.$$

Proof. First we have

$$(2.5) \qquad \begin{aligned} \frac{\partial \mathcal{L}}{\partial t_n} &= \frac{\partial \mathcal{B}_n}{\partial k} \frac{\partial \mathcal{L}}{\partial x} - \frac{\partial \mathcal{B}_n}{\partial x} \frac{\partial \mathcal{L}}{\partial k} \\ &= \frac{\partial \mathcal{B}_n}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial k} \frac{\partial \mathcal{L}}{\partial x} - \left(\frac{\partial \mathcal{B}_n}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial x} + \frac{\partial \mathcal{B}_n}{\partial x} \Big|_{\mathcal{L} \text{ fixed}} \right) \frac{\partial \mathcal{L}}{\partial k} \\ &= - \frac{\partial \mathcal{B}_n}{\partial x} \Big|_{\mathcal{L} \text{ fixed}} \frac{\partial \mathcal{L}}{\partial k}. \end{aligned}$$

The formulas

$$\{\mathcal{L}, \mathcal{M}\} = 1 \qquad \text{and} \qquad \frac{\partial \mathcal{M}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{M}\}$$

satisfied by $\mathcal{M}$ give us

$$\begin{aligned} \frac{\partial \mathcal{M}}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial t_n} + \frac{\partial \mathcal{M}}{\partial t_n} \Big|_{\mathcal{L} \text{ fixed}} &= \frac{\partial \mathcal{B}_n}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial k} \frac{\partial \mathcal{M}}{\partial x} - \left(\frac{\partial \mathcal{B}_n}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial x} + \frac{\partial \mathcal{B}_n}{\partial x} \Big|_{\mathcal{L} \text{ fixed}} \right) \frac{\partial \mathcal{M}}{\partial k} \\ &= \frac{\partial \mathcal{B}_n}{\partial \mathcal{L}} - \frac{\partial \mathcal{B}_n}{\partial x} \Big|_{\mathcal{L} \text{ fixed}} \frac{\partial \mathcal{M}}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial k}. \end{aligned}$$

In view of (2.5), we have

$$\frac{\partial \mathcal{M}}{\partial t_n} \Big|_{\mathcal{L} \text{ fixed}} = \frac{\partial \mathcal{B}_n}{\partial \mathcal{L}} = n\mathcal{L}^{n-1} + \sum_{m=1}^{\infty} m a_{n,m} \mathcal{L}^{-m-1}.$$

Comparing coefficients give us the assertion. The proof for the formula for $\log \mathcal{P}$ is the same, with $\log \mathcal{P}$ replacing $\mathcal{B}_n$, s replacing t_n. $\qquad \square$

The proof of this proposition also gives us a characterization of the function $\mathcal{M}$ as follows.

Proposition 2.4. *If $\mathcal{M} = \sum_{n=1}^{\infty} n\mathcal{L}^{n-1} + x + \frac{Ns}{\mathcal{L}} + \sum_{n=1}^{\infty} v_n \mathcal{L}^{-n-1}$ is a function such that*

$$\{\mathcal{L}, \mathcal{M}\} = 1, \qquad \frac{\partial \mathcal{M}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{M}\}, \qquad \frac{\partial \mathcal{M}}{\partial s} = \{\log \mathcal{P}, \mathcal{M}\},$$

then there exists a dressing function φ as in Proposition 2.2 and satisfies

$$\mathcal{M} = e^{\mathrm{ad}\,\varphi} \left(\sum_{n=1}^{\infty} n k^{n-1} + x + \frac{Ns}{k} \right).$$

In other words, $\mathcal{M}$ is an Orlov function of $\mathcal{L}$.

Proof. We let φ' be a dressing function given by Proposition 2.2 and define

$$\begin{aligned} \mathcal{M}' &= e^{\mathrm{ad}\,\varphi'} \left(\sum_{n=1}^{\infty} n t_n k^{n-1} + x + \frac{Ns}{k} \right) \\ &= \sum_{n=1}^{\infty} n\mathcal{L}^{n-1} + x + \frac{Ns}{\mathcal{L}} + \sum_{n=1}^{\infty} v'_n \mathcal{L}^{-n-1}. \end{aligned}$$

Then we have

$$\{\mathcal{L}, \mathcal{M}'\} = 1, \qquad \frac{\partial \mathcal{M}'}{\partial t_n} = \{\mathcal{B}_n, \mathcal{M}'\}, \qquad \frac{\partial \mathcal{M}'}{\partial s} = \{\log \mathcal{P}, \mathcal{M}'\}.$$

From the proof in the previous proposition, we see that these conditions imply that

$$(2.6) \qquad \frac{\partial \mathcal{M}'}{\partial t_n}\bigg|_{\mathcal{L} \text{ fixed}} = \frac{\partial \mathcal{B}_n}{\partial \mathcal{L}}, \qquad \frac{\partial \mathcal{M}}{\partial t_n}\bigg|_{\mathcal{L} \text{ fixed}} = \frac{\partial \mathcal{B}_n}{\partial \mathcal{L}},$$

$$(2.7) \qquad \frac{\partial \mathcal{M}'}{\partial s}\bigg|_{\mathcal{L} \text{ fixed}} = \frac{\partial \log \mathcal{P}}{\partial \mathcal{L}}, \qquad \frac{\partial \mathcal{M}}{\partial s}\bigg|_{\mathcal{L} \text{ fixed}} = \frac{\partial \log \mathcal{P}}{\partial \mathcal{L}}.$$

From the explicit expressions of $\mathcal{M}$ and $\mathcal{M}'$, we have

$$\mathcal{M}' - \mathcal{M} = \sum_{m=1}^{\infty} c_m \mathcal{L}^{-m-1}.$$

The equations in (2.6) imply that the c_m's are constants. We let $\varphi_0 = \sum_{m=1}^{\infty} \frac{c_m}{m} k^{-m}$, then

$$e^{\text{ad}\,\varphi_0}\left(\sum_{n=1}^{\infty} nt_n k^{n-1} + x + \frac{Ns}{k}\right) = \sum_{n=1}^{\infty} nt_n k^{n-1} + x + \frac{Ns}{k} - \sum_{m=1}^{\infty} c_m k^{-m-1}.$$

Hence if we define φ by $e^{\text{ad}\,\varphi} = e^{\text{ad}\,\varphi'} e^{\text{ad}\,\varphi_0}$, then

$$e^{\text{ad}\,\varphi}\left(\sum_{n=1}^{\infty} nt_n k^{n-1} + x + \frac{Ns}{k}\right) = e^{\text{ad}\,\varphi'}\left(\sum_{n=1}^{\infty} nt_n k^{n-1} + x + \frac{Ns}{k} - \sum_{m=1}^{\infty} c_m k^{-m-1}\right)$$

$$= \mathcal{M}' - \sum_{m=1}^{\infty} c_m \mathcal{L}^{-m-1} = \mathcal{M}.$$

Since φ_0 has constant coefficients, φ satisfies all the requirements in Proposition 2.2. $\qquad\square$

Now we introduce the fundamental two form

$$\omega = dk \wedge dx + \sum_{n=1}^{\infty} d\mathcal{B}_n \wedge dt_n + d\left(\log \mathcal{P} - \frac{\partial \phi}{\partial s}\right) \wedge ds.$$

The exterior derivative d is taken with respect to the independent variables k, x, s and $t_n, n = 1, 2, \ldots$. It satisfies

$$d\omega = 0, \qquad \text{and} \qquad \omega \wedge \omega = 0.$$

In fact, $(\mathcal{L}, \mathcal{M})$ is a pair of functions that play the role of Darboux coordinates. Namely

$$d\mathcal{L} \wedge d\mathcal{M} = \omega.$$

Proposition 2.5. *The system of equations* (2.1) *and* (2.4), *together with* $\{\mathcal{L}, \mathcal{M}\} = 1$ *are equivalent to*

$$(2.8) \qquad d\mathcal{L} \wedge d\mathcal{M} = dk \wedge dx + \sum_{n=1}^{\infty} d\mathcal{B}_n \wedge dt_n + d(\log \mathcal{P} - \frac{\partial \phi}{\partial s}) \wedge ds.$$

Proof. We only show that (2.8) implies (2.1) and (2.4), together with $\{\mathcal{L}, \mathcal{M}\} = 1$. The other direction follows by tracing the argument backward. By looking at the coefficients of $dk \wedge dx$, $dk \wedge dt_n$, $dx \wedge dt_n$, $dk \wedge ds$, $dx \wedge ds$ and $dt_n \wedge ds$ respectively and using the property of ϕ given by Proposition 2.1, we find that (2.8) gives

$$\frac{\partial \mathcal{L}}{\partial k}\frac{\partial \mathcal{M}}{\partial x} - \frac{\partial \mathcal{L}}{\partial x}\frac{\partial \mathcal{M}}{\partial k} = \{\mathcal{L}, \mathcal{M}\} = 1,$$

$$\frac{\partial \mathcal{L}}{\partial k}\frac{\partial \mathcal{M}}{\partial t_n} - \frac{\partial \mathcal{M}}{\partial k}\frac{\partial \mathcal{L}}{\partial t_n} = \frac{\partial B_n}{\partial k}, \qquad \frac{\partial \mathcal{L}}{\partial x}\frac{\partial \mathcal{M}}{\partial t_n} - \frac{\partial \mathcal{M}}{\partial x}\frac{\partial \mathcal{L}}{\partial t_n} = \frac{\partial B_n}{\partial x},$$

$$\frac{\partial \mathcal{L}}{\partial k}\frac{\partial \mathcal{M}}{\partial s} - \frac{\partial \mathcal{M}}{\partial k}\frac{\partial \mathcal{L}}{\partial s} = \frac{\partial \log \mathcal{P}}{\partial k}, \qquad \frac{\partial \mathcal{L}}{\partial x}\frac{\partial \mathcal{M}}{\partial s} - \frac{\partial \mathcal{M}}{\partial x}\frac{\partial \mathcal{L}}{\partial s} = \frac{\partial \log \mathcal{P}}{\partial x},$$

$$\frac{\partial \mathcal{L}}{\partial s}\frac{\partial \mathcal{M}}{\partial t_n} - \frac{\partial \mathcal{M}}{\partial s}\frac{\partial \mathcal{L}}{\partial t_n} = -\frac{\partial \log \mathcal{P}}{\partial t_n} + \frac{\partial^2 \phi}{\partial s \partial t_n} + \frac{\partial \mathcal{B}_n}{\partial s} = -\frac{\partial \log \mathcal{P}}{\partial t_n} + \frac{\partial (\mathcal{L}^n)_{\geq 0}}{\partial s}.$$

The equations in the second line combine to give

$$\begin{pmatrix} -\frac{\partial \mathcal{M}}{\partial k} & \frac{\partial \mathcal{L}}{\partial k} \\ -\frac{\partial \mathcal{M}}{\partial x} & \frac{\partial \mathcal{L}}{\partial x} \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial t_n} \\ \frac{\partial \mathcal{M}}{\partial t_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{B}_n}{\partial k} \\ \frac{\partial \mathcal{B}_n}{\partial x} \end{pmatrix},$$

or since $\{\mathcal{L}, \mathcal{M}\} = 1$

$$\begin{pmatrix} \frac{\partial \mathcal{L}}{\partial t_n} \\ \frac{\partial \mathcal{M}}{\partial t_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial x} & -\frac{\partial \mathcal{L}}{\partial k} \\ \frac{\partial \mathcal{M}}{\partial x} & -\frac{\partial \mathcal{M}}{\partial k} \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{B}_n}{\partial k} \\ \frac{\partial \mathcal{B}_n}{\partial x} \end{pmatrix}$$

i.e.

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \qquad \frac{\partial \mathcal{M}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{M}\}.$$

Similarly, the equations in the third line combine to give

$$\frac{\partial \mathcal{L}}{\partial s} = \{\log \mathcal{P}, \mathcal{L}\}, \qquad \frac{\partial \mathcal{M}}{\partial s} = \{\log \mathcal{P}, \mathcal{M}\}.$$

Using the previous result, the last equation gives

$$-\frac{\partial \log \mathcal{P}}{\partial t_n} + \frac{\partial (\mathcal{L}^n)_{\geq 0}}{\partial s} = \left(\frac{\partial \log \mathcal{P}}{\partial k}\frac{\partial \mathcal{L}}{\partial x} - \frac{\partial \log \mathcal{P}}{\partial x}\frac{\partial \mathcal{L}}{\partial k} \right) \left(\frac{\partial \mathcal{B}_n}{\partial k}\frac{\partial \mathcal{M}}{\partial x} - \frac{\partial \mathcal{B}_n}{\partial x}\frac{\partial \mathcal{M}}{\partial k} \right)$$

$$- \left(\frac{\partial \log \mathcal{P}}{\partial k}\frac{\partial \mathcal{M}}{\partial x} - \frac{\partial \log \mathcal{P}}{\partial x}\frac{\partial \mathcal{M}}{\partial k} \right) \left(\frac{\partial \mathcal{B}_n}{\partial k}\frac{\partial \mathcal{L}}{\partial x} - \frac{\partial \mathcal{B}_n}{\partial x}\frac{\partial \mathcal{L}}{\partial k} \right)$$

$$= \{\log \mathcal{P}, \mathcal{B}_n\}.$$

The coefficients of $dt_n \wedge dt_m$ gives the zero curvature condition. $\qquad \square$

2.4. **S function and tau function.** Proposition 2.5 implies that we can find a function S such that

$$dS = \mathcal{M}d\mathcal{L} + kdx + \sum_{n=1}^{\infty} \mathcal{B}_n dt_n + (\log \mathcal{P} - \frac{\partial \phi}{\partial s})ds.$$

In other words,

$$\frac{\partial S}{\partial \mathcal{L}} = \mathcal{M}, \quad \frac{\partial S}{\partial x}\Big|_{\mathcal{L} \text{ fixed}} = k, \quad \frac{\partial S}{\partial t_n}\Big|_{\mathcal{L} \text{ fixed}} = \mathcal{B}_n, \quad \frac{\partial S}{\partial s}\Big|_{\mathcal{L} \text{ fixed}} = \log \mathcal{P} - \frac{\partial \phi}{\partial s}.$$

The second equation is just a special case of the third when $n = 1$ since $\mathcal{B}_1 = k$. From the explicit representation of the function $\mathcal{B}_n$'s and $\log \mathcal{P}$ given by Proposition 2.3, we have the following explicit expression for S.

Proposition 2.6.

$$S = \sum_{n=1}^{\infty} t_n \mathcal{L}^n + x\mathcal{L} + s \log \mathcal{L}^N + \sum_{n=1}^{\infty} -\frac{v_n}{n}\mathcal{L}^{-n} - \phi.$$

We also have the following characterization of the partial derivatives of v_n's in terms of residues.

Proposition 2.7.

$$\frac{\partial v_n}{\partial t_m} = \operatorname{Res} \mathcal{L}^n d_k \mathcal{B}_m, \qquad \frac{\partial v_n}{\partial s} = \operatorname{Res} \mathcal{L}^n d_k \log \mathcal{P}.$$

Here the differential d_k is taken with respect to k and $\operatorname{Res} A dk$ means the coefficient of k^{-1} of A.

Proof. We give a proof which follows the same line as Proposition 4 in [TT92]. We only show the second equality here. We have

$$\frac{\partial \mathcal{M}}{\partial s} - \frac{\partial \mathcal{M}}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial s} = \frac{\partial \mathcal{M}}{\partial s}\Big|_{\mathcal{L} \text{ fixed}} = \frac{N}{\mathcal{L}} + \sum_{n=1}^{\infty} \frac{\partial v_n}{\partial s}\mathcal{L}^{-n-1}.$$

Using $\operatorname{Res} \mathcal{L}^n d_k \mathcal{L} = \delta_{n,-1}$, we have

$$\frac{\partial v_n}{\partial s} = \operatorname{Res} \mathcal{L}^n \left(\frac{\partial \mathcal{M}}{\partial s} - \frac{\partial \mathcal{M}}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial s}\right) d_k \mathcal{L}.$$

The expression in the bracket

$$\left(\frac{\partial \mathcal{M}}{\partial s} - \frac{\partial \mathcal{M}}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial s}\right) d_k \mathcal{L}$$

$$= \left(\{\log \mathcal{P}, \mathcal{M}\}\frac{\partial \mathcal{L}}{\partial k} - \{\log \mathcal{P}, \mathcal{L}\}\frac{\partial \mathcal{M}}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial k}\right) dk$$

$$= \left(\left(\frac{\partial \log \mathcal{P}}{\partial k}\frac{\partial \mathcal{M}}{\partial x} - \frac{\partial \log \mathcal{P}}{\partial x}\frac{\partial \mathcal{M}}{\partial k}\right)\frac{\partial \mathcal{L}}{\partial k} - \left(\frac{\partial \log \mathcal{P}}{\partial k}\frac{\partial \mathcal{L}}{\partial x} - \frac{\partial \log \mathcal{P}}{\partial x}\frac{\partial \mathcal{L}}{\partial k}\right)\frac{\partial \mathcal{M}}{\partial k}\right) dk$$

$$= \frac{\partial \log \mathcal{P}}{\partial k}dk = d_k \log \mathcal{P}.$$

This implies the assertion.   □

As a consequence of this proposition, we can show the existence of a tau function for our dcmKP hierarchy. First we have the following properties of taking residues,

$$\operatorname{Res} A d_k B = - \operatorname{Res} B d_k A,$$
$$\operatorname{Res} A d_k B = \operatorname{Res} A_{>0} d_k B_{<0} + \operatorname{Res} A_{<0} d_k B_{>0}$$

When $m, n \geq 1$, we have $\operatorname{Res} \mathcal{L}^m d_k \mathcal{L}^n = n \operatorname{Res} \mathcal{L}^{m+n-1} d_k \mathcal{L} = 0$. Hence

$$\operatorname{Res}(\mathcal{L}^m)_{>0} d_k (\mathcal{L}^n)_{<0} = - \operatorname{Res}(\mathcal{L}^m)_{<0} d_k (\mathcal{L}^n)_{>0}.$$

It follows that

(2.9)
$$\frac{\partial v_m}{\partial t_n} = \operatorname{Res} \mathcal{L}^m d_k \mathcal{B}_n = \operatorname{Res}(\mathcal{L}^m)_{<0} d_k (\mathcal{L}_n)_{>0} = - \operatorname{Res}(\mathcal{L}^m)_{>0} d_k (\mathcal{L}^n)_{<0}$$
$$= \operatorname{Res}(\mathcal{L}^n)_{<0} d_k (\mathcal{L}^m)_{>0} = \frac{\partial v_n}{\partial t_m}.$$

On the other hand, since

$$\frac{\partial}{\partial t_m} \operatorname{Res} \mathcal{L}^n d_k \log \mathcal{P} = \frac{\partial^2 v_n}{\partial s \partial t_m} = \frac{\partial^2 v_m}{\partial s \partial t_n} = \frac{\partial}{\partial t_n} \operatorname{Res} \mathcal{L}^m d_k \log \mathcal{P},$$

there exists a function $\Phi(t)$ such that

(2.10) $$\frac{\partial \Phi}{\partial t_n} = \operatorname{Res} \mathcal{L}^n d_k \log \mathcal{P} = \frac{\partial v_n}{\partial s}.$$

Φ is only determined up to a function of s. Finally from (2.9) and (2.10), we have

Proposition 2.8. *There exist a function τ of t, called the tau function of our dcmKP hierarchy, such that $\log \tau$ is the generating function for v_n's and Φ, i.e.*

$$\frac{\partial \log \tau}{\partial t_n} = v_n, \qquad\qquad \frac{\partial \log \tau}{\partial s} = \Phi.$$

Since Φ is only determined up to a function of s, τ is also only determined up to a function of s [2].

Remark 2.9. In the special case when $\mathcal{P} = k$, we have

$$\operatorname{Res} \mathcal{L}^n d_k \log k = (\mathcal{L}^n)_0.$$

Hence the function Φ coincides with the function ϕ we introduce in Proposition 2.1.

---

[2]This degree of freedom can also be viewed as due to we do not specify the s flow of $\mathcal{P}$.

In this case, the formulas in Proposition 2.3 can be rewritten in terms of the tau function as

$$(2.11) \qquad \mathcal{B}_n = \mathcal{L}^n - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^2 \log \tau}{\partial t_m \partial t_n} \mathcal{L}^{-m} - \frac{\partial^2 \log \tau}{\partial s \partial t_n},$$

$$\log k = \log \mathcal{L} - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^2 \log \tau}{\partial t_m \partial s} \mathcal{L}^{-m}.$$

Hence the coefficients of $\mathcal{L}, \mathcal{M}, \mathcal{B}_n$ can be expressed as differential polynomials of derivatives of $\log \tau$. We shall see below that this special case play a particular role of bridging the transition from dispersionless Toda (dToda) hierarchy to dispersionless KP (dKP) hierarchy. This is the motivation for our present work.

3. Relations with dispersionless Toda hierarchy

There are a few ways to formulate dispersionless Toda hierarchy, All of them are connected by a Miura type transformation. We first quickly review the set up following [TT95]. For details, we refer to [TT95] and references therein.

dToda is a system of differential equations with two sets of independent variables $(t_1, t_2, \dots), (t_{-1}, t_{-2}, \dots)$ and an independent variable s. The fundamental quantities are two formal power series

$$\mathcal{L}(p) = p + \sum_{n=0}^{\infty} u_{n+1}(t) p^{-n}$$

$$\tilde{\mathcal{L}}^{-1}(p) = \tilde{u}_0(t) p^{-1} + \sum_{n=0}^{\infty} \tilde{u}_{n+1}(t) p^n.$$

Here t denotes collectively all the independent variables. The Lax representation is

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{(\mathcal{L}^n)_{\geq 0}, \mathcal{L}\}_T, \qquad\qquad \frac{\partial \mathcal{L}}{\partial t_{-n}} = \{(\tilde{\mathcal{L}}^{-n})_{<0}, \mathcal{L}\}_T,$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial t_n} = \{(\mathcal{L}^n)_{\geq 0}, \tilde{\mathcal{L}}\}_T, \qquad\qquad \frac{\partial \tilde{\mathcal{L}}}{\partial t_{-n}} = \{(\tilde{\mathcal{L}}^{-n})_{<0}, \tilde{\mathcal{L}}\}_T.$$

Here $\{\cdot, \cdot\}_T$ is the Poisson bracket for dToda hierarchy

$$\{f, g\}_T = p \frac{\partial f}{\partial p} \frac{\partial g}{\partial s} - p \frac{\partial f}{\partial s} \frac{\partial g}{\partial p}.$$

In order to see the relation between the dToda hierarchy and our dcmKP hierarchy, we use the fact that (for details see [TT95]) there exist a tau

function τ_{dToda} such that in terms of this tau function,

$$(3.1) \qquad (\mathcal{L}^n)_{\geq 0} = \mathcal{L}^n - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^2 \log \tau_{\mathrm{dToda}}}{\partial t_m \partial t_n} \mathcal{L}^{-m},$$

$$(\mathcal{L}^n)_0 = \frac{\partial^2 \log \tau_{\mathrm{dToda}}}{\partial s \partial t_n},$$

$$\log p = \log \mathcal{L} - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^2 \log \tau_{\mathrm{dToda}}}{\partial t_m \partial s} \mathcal{L}^{-m}.$$

Comparing with equations (2.11), it is natural to see that the following proposition should hold.

Proposition 3.1. *If $(\mathcal{L}, \tilde{\mathcal{L}})$ is a solution to dToda and $\frac{\partial(\mathcal{L})_0}{\partial s} = 0$, then $(\mathcal{L}, \mathcal{P} = k)$ is also a solution to dcmKP hierarchy if we replace p by k, t_1 by $t_1 + x$ and regarding t_{-n}'s as parameters. In this case, the tau function for the dToda hierarchy τ_{dToda} is also the tau function for the corresponding dcmKP hierarchy.*

Proof. Replacing p by k and t_1 by $t_1 + x$, the case $n = 1$ of the first equation in the dToda hierarchy gives us

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial t_1} = k \frac{\partial(\mathcal{L})_{\geq 0}}{\partial k} \frac{\partial \mathcal{L}}{\partial s} - k \frac{\partial(\mathcal{L})_{\geq 0}}{\partial s} \frac{\partial \mathcal{L}}{\partial k}.$$

Now $(\mathcal{L})_{\geq 0} = k + u_1$. Since we assume that u_1 is independent of s, $\frac{\partial(\mathcal{L})_{\geq 0}}{\partial s}$ is identically 0. Hence

$$\frac{\partial \mathcal{L}}{\partial x} = k \frac{\partial \mathcal{L}}{\partial s},$$

or equivalently,

$$\frac{\partial \mathcal{L}}{\partial s} = \{\log k, \mathcal{L}\} = \frac{1}{k} \frac{\partial \mathcal{L}}{\partial x},$$

which is the second equation in our dcmKP hierarchy (2.1) with $\mathcal{P} = k$.
   From this equation, we also have

$$\frac{\partial \mathcal{L}^n}{\partial x} = k \frac{\partial \mathcal{L}^n}{\partial s}.$$

Comparing powers of k, we have

$$\frac{\partial(\mathcal{L}^n)_{>0}}{\partial x} = k \frac{\partial(\mathcal{L}^n)_{\geq 0}}{\partial s},$$

or equivalently,

$$\frac{\partial(\mathcal{L}^n)_{\geq 0}}{\partial s} = \frac{1}{k} \frac{\partial(\mathcal{L}^n)_{>0}}{\partial x} = \{\log k, (\mathcal{L}^n)_{>0}\},$$

which is the third equation in our dcmKP hierarchy (2.1). Finally using the other equations in the dToda hierarchy, we have

$$\frac{\partial \mathcal{L}}{\partial t_n} = k\frac{\partial (\mathcal{L}^n)_{\geq 0}}{\partial k}\frac{\partial \mathcal{L}}{\partial s} - k\frac{\partial (\mathcal{L}^n)_{\geq 0}}{\partial s}\frac{\partial \mathcal{L}}{\partial k} = \frac{\partial (\mathcal{L}^n)_{\geq 0}}{\partial k}\frac{\partial \mathcal{L}}{\partial x} - \frac{\partial (\mathcal{L}^n)_{> 0}}{\partial x}\frac{\partial \mathcal{L}}{\partial k}$$

$$= \frac{\partial (\mathcal{L}^n)_{> 0}}{\partial k}\frac{\partial \mathcal{L}}{\partial x} - \frac{\partial (\mathcal{L}^n)_{> 0}}{\partial x}\frac{\partial \mathcal{L}}{\partial k} = \{(\mathcal{L}^n)_{> 0}, \mathcal{L}\},$$

which is the first equation in our dcmKP hierarchy (2.1). $\square$

Remark 3.2. In the proof of this proposition, we also see that in the special case when $\mathcal{P} = k$, the third equation in (2.1) is a consequence of the second equation.

4. Miura map between dmKP and dKP hierarchies

We establish that if $(\mathcal{L}, \mathcal{P})$ is a solution of our dcmKP hierarchy, then via a Miura transform, it will give a solution of the dmKP hierarchy defined by Takebe [Tak02]. The fundamental quantity in Takebe's definition is the formal series

$$\tilde{\mathcal{L}} = k + \sum_{n=1}^{\infty} \tilde{u}_{n+1}(t)k^{-n},$$

and an auxiliary polynomial

$$\tilde{\mathcal{P}} = k^N + \tilde{p}_{N-1}(t)k^{N-1} + \ldots + \tilde{p}_0(t).$$

Here $t = (x, s, t_1, t_2, \dots)$ are independent variables. The Lax representation is

(4.1)
$$\frac{\partial \tilde{\mathcal{L}}}{\partial t_n} = \{(\tilde{\mathcal{L}}^n)_{\geq 0}, \tilde{\mathcal{L}}\}, \qquad \frac{\partial \tilde{\mathcal{L}}}{\partial s} = \{\log \tilde{\mathcal{P}}, \tilde{\mathcal{L}}\},$$

$$\frac{\partial \log \tilde{\mathcal{P}}}{\partial t_n} = \frac{\partial (\tilde{\mathcal{L}}^n)_{\geq 0}}{\partial s} - \{\log \tilde{\mathcal{P}}, (\tilde{\mathcal{L}}^n)_{\geq 0}\}.$$

Let $(\mathcal{L}, \mathcal{P})$ be a solution of our dcmKP hierarchy (2.1) and ϕ be the function defined by Proposition 2.1. The Miura transform of $(\mathcal{L}, \mathcal{P})$ is given by

$$\tilde{\mathcal{L}} = e^{\mathrm{ad}\,\phi}\mathcal{L}, \qquad \tilde{\mathcal{P}} = e^{\mathrm{ad}\,\phi}\mathcal{P}.$$

Since ϕ is independent of k, we find

$$e^{\mathrm{ad}\,\phi}\mathcal{L} = e^{\mathrm{ad}\,\phi}k + \sum_{n=0}^{\infty} u_{n+1}(t)(e^{\mathrm{ad}\,\phi}k)^{-n},$$

and

$$e^{\mathrm{ad}\,\phi}k = k - \frac{\partial \phi}{\partial x} = k - u_1.$$

Hence

$$\tilde{\mathcal{L}} = k - u_1(t) + \sum_{n=0}^{\infty} u_{n+1}(t)(k - u_1(t))^{-n} = k + \sum_{n=1}^{\infty} \tilde{u}_{n+1}(t)k^{-n}$$

does not has term in k^0.

Proposition 4.1. *Let $(\mathcal{L}, \mathcal{P})$ be a solution of the dcmKP hierarchy (2.1), then $(\tilde{\mathcal{L}}, \tilde{\mathcal{P}})$ defined by the Miura transform is a solution of the dmKP hierarchy defined by Takebe [Tak02].*

Proof. We prove that $(\tilde{\mathcal{L}}, \tilde{\mathcal{P}})$ satisfies the dmKP hierarchy (4.1) defined by Takebe. Using the formulas in Appendix A of [TT95], we have

$$\begin{aligned}
\frac{\partial \tilde{\mathcal{L}}}{\partial t_n} &= e^{\operatorname{ad}\phi}\frac{\partial \mathcal{L}}{\partial t_n} + \{\nabla_{t_n,\phi}\phi, \tilde{\mathcal{L}}\} \\
&= \{e^{\operatorname{ad}\phi}(\mathcal{L}^n)_{>0}, e^{\operatorname{ad}\phi}\mathcal{L}\} + \{\frac{\partial\phi}{\partial t_n}, \tilde{\mathcal{L}}\}
\end{aligned}$$

Now if we write

$$\mathcal{L}^n = \sum_{m=-\infty}^{n} v_{n,m}(t)k^m$$

we have

$$\tilde{\mathcal{L}}^n = \sum_{m=-\infty}^{n} v_{n,m}(t)(k-u_1)^m$$

Hence
(4.2)
$$(\tilde{\mathcal{L}}^n)_{\geq 0} = \sum_{m=0}^{n} v_{n,m}(k-u_1)^m = \sum_{m=1}^{\infty} v_{n,m}(k-u_1)^m + (\mathcal{L}^n)_0 = e^{\operatorname{ad}\phi}(\mathcal{L}^n)_{>0} + \frac{\partial\phi}{\partial t_n}.$$

Hence we have established the first equation in (4.1). Similarly, we have

$$\begin{aligned}
\frac{\partial \tilde{\mathcal{L}}}{\partial s} &= e^{\operatorname{ad}\phi}\frac{\partial \mathcal{L}}{\partial s} + \{\nabla_{s,\phi}\phi, \tilde{\mathcal{L}}\} \\
&= \{e^{\operatorname{ad}\phi}\log\mathcal{P}, e^{\operatorname{ad}\phi}\mathcal{L}\} + \{\frac{\partial\phi}{\partial s}, \tilde{\mathcal{L}}\}.
\end{aligned}$$

Since $\frac{\partial\phi}{\partial s}$ is independent of k and x, we have established the second equation in (2.1). Finally,

$$\begin{aligned}
\frac{\partial \log\tilde{\mathcal{P}}}{\partial t_n} &= e^{\operatorname{ad}\phi}\frac{\partial \log\mathcal{P}}{\partial t_n} + \{\nabla_{t_n,\phi}\phi, \log\tilde{\mathcal{P}}\} \\
&= e^{\operatorname{ad}\phi}\left(\frac{\partial(\mathcal{L}^n)_{\geq 0}}{\partial s} - \{\log\mathcal{P}, (\mathcal{L}^n)_{>0}\}\right) + \{(\mathcal{L}^n)_0, \log\tilde{\mathcal{P}}\} \\
&= \frac{\partial(\tilde{\mathcal{L}}^n)_{\geq 0}}{\partial s} - \{\log\tilde{\mathcal{P}}, e^{\operatorname{ad}\phi}(\mathcal{L}^n)_{>0}\} - \{\log\tilde{\mathcal{P}}, (\mathcal{L}^n)_0\} \\
&= \frac{\partial(\tilde{\mathcal{L}}^n)_{\geq 0}}{\partial s} - \{\log\tilde{\mathcal{P}}, (\tilde{\mathcal{L}}^n)_{\geq 0}\}.
\end{aligned}$$

which is the third equation in (4.1). □

Notice that if we regard s as a parameter, then $\tilde{\mathcal{L}}$ is a solution of the dKP hierarchy.

Remark 4.2. To go in the opposite direction, we have

$$(4.3) \qquad\qquad \mathcal{L} = e^{-\operatorname{ad}\phi}\tilde{\mathcal{L}}.$$

Express in terms of $\tilde{\mathcal{L}}$, we have from (4.2),

$$\frac{\partial\phi}{\partial t_n} = (\mathcal{L}^n)_0 = (\tilde{\mathcal{L}}^n)_{\geq 0}\Big|_{k=u_1} = (\tilde{\mathcal{L}}^n)_{\geq 0}\Big|_{k=\frac{\partial\phi}{\partial x}}.$$

This is precisely the condition that ϕ must satisfy in order that (4.3) transforms a dKP solution to a dmKP solution which is proved by Chang and Tu in [CT00].

Remark 4.3. Notice that if φ is the dressing operator for $\mathcal{L}$, then the function $\tilde{\varphi}$ defined by $e^{\operatorname{ad}\tilde{\varphi}} = e^{\operatorname{ad}\phi}e^{\operatorname{ad}\varphi}$ is the dressing operator for $\tilde{\mathcal{L}}$. If we denote the Orlov function for $\tilde{\mathcal{L}}$ by $\tilde{\mathcal{M}}$, then by definition (see [Tak02])

$$
\begin{aligned}
\tilde{\mathcal{M}} =& e^{\operatorname{ad}\tilde{\varphi}}\left(\sum_{n=1}^{\infty} nt_n k^{n-1} + x + \frac{Ns}{k}\right) = e^{\operatorname{ad}\phi}e^{\operatorname{ad}\varphi}\left(\sum_{n=1}^{\infty} nt_n k^{n-1} + x + \frac{Ns}{k}\right) \\
=& e^{\operatorname{ad}\phi}\left(\sum_{n=1}^{\infty} nt_n \mathcal{L}^{n-1} + x + \frac{Ns}{\mathcal{L}} + \sum_{n=1}^{\infty} v_n \mathcal{L}^{-n-1}\right) \\
=& \sum_{n=1}^{\infty} nt_n \tilde{\mathcal{L}}^{n-1} + x + \frac{Ns}{\tilde{\mathcal{L}}} + \sum_{n=1}^{\infty} v_n \tilde{\mathcal{L}}^{-n-1}.
\end{aligned}
$$

Notice that the functions v_n's are the same in both hierarchies. In particular, the tau function for a solution of our dcmKP hierarchy is also the tau function for the corresponding dmKP hierarchy obtained via a Miura transform.

5. Twistor construction

As in [TT95] and [Tak02], we show that a solution of dcmKP can be obtained from twistor construction (or Riemann Hilbert type construction).

A twistor data for a solution $(\mathcal{L}, \mathcal{P})$ of dcmKP hierarchy is a pair of functions (f, g) in variables (k, x) such that $\{f, g\} = 1$ and

$$f(\mathcal{L}, \mathcal{M})_{\leq 0} = 0, \qquad\qquad g(\mathcal{L}, \mathcal{M})_{< 0} = 0.$$

Proposition 5.1. *Let (f, g) be a pair of functions in (k, x) given such that $\{f, g\} = 1$. Let*

$$\mathcal{L} = k + \sum_{n=0}^{\infty} u_{n+1} k^{-n},$$

$$\mathcal{M} = \sum_{n=1}^{\infty} n t_n \mathcal{L}^{n-1} + x + \frac{Ns}{\mathcal{L}} + \sum_{n=1}^{\infty} v_n \mathcal{L}^{-n-1},$$

$$\mathcal{P} = k^N + p_{N-1}(t) k^{N-1} + \ldots + p_0(t)$$

be formal power series in k with coefficients depending on t. If

(5.1) $f(\mathcal{L}, \mathcal{M})_{\leq 0} = 0,$ $g(\mathcal{L}, \mathcal{M})_{<0} = 0,$

   $(\{\log \mathcal{P}, f(\mathcal{L}, \mathcal{M})\})_{\leq 0} = 0,$ $(\{\log \mathcal{P}, g(\mathcal{L}, \mathcal{M})\})_{<-1} = 0,$

Then $(\mathcal{L}, \mathcal{P})$ is a solution to our dcmKP hierarchy with $\mathcal{M}$ the corresponding Orlov function.

Proof. For the case when $\mathcal{L}$ is independent of s, the first two conditions are those given by Chang and Tu [CT00] in order that $\mathcal{L}, \mathcal{M}$ satisfies $\{\mathcal{L}, \mathcal{M}\} = 1$, the first equations in (2.1) and the first equations in (2.4) of the dcmKP hierarchy. The other two conditions (which are modified by those given by Takebe in [Tak02][3]) are such that the other equations in (2.1) and (2.4) are satisfied. For completeness, we give the full proof here.

   We let

(5.2) $\widehat{\mathcal{L}} = f(\mathcal{L}, \mathcal{M})$ and $\widehat{\mathcal{M}} = g(\mathcal{L}, \mathcal{M}).$

Then the condition of the proposition says that

$$\widehat{\mathcal{L}}_{\leq 0} = 0, \quad \widehat{\mathcal{M}}_{<0} = 0, \quad \{\log \mathcal{P}, \widehat{\mathcal{L}}\}_{\leq 0} = 0, \quad \{\log \mathcal{P}, \widehat{\mathcal{M}}\}_{<-1} = 0,$$

Taking derivative of the equations in (5.2) with respect to k and x, we have the following system:

(5.3) $\begin{pmatrix} \frac{\partial f}{\partial \mathcal{L}} & \frac{\partial f}{\partial \mathcal{M}} \\ \frac{\partial g}{\partial \mathcal{L}} & \frac{\partial g}{\partial \mathcal{M}} \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial k} & \frac{\partial \mathcal{L}}{\partial x} \\ \frac{\partial \mathcal{M}}{\partial k} & \frac{\partial \mathcal{M}}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial \widehat{\mathcal{L}}}{\partial k} & \frac{\partial \widehat{\mathcal{L}}}{\partial x} \\ \frac{\partial \widehat{\mathcal{M}}}{\partial k} & \frac{\partial \widehat{\mathcal{M}}}{\partial x} \end{pmatrix}$

Taking determinant of both sides, since $\{f, g\} = 1$, we have

$$\{\mathcal{L}, \mathcal{M}\} = \{\widehat{\mathcal{L}}, \widehat{\mathcal{M}}\}.$$

---

[3]Notice that our f and g do not depend on s, though in general we can let them have one more degree of freedom in s.

Now the left hand side

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial k}\frac{\partial \mathcal{M}}{\partial x} - \frac{\partial \mathcal{L}}{\partial x}\frac{\partial \mathcal{M}}{\partial k} &= \frac{\partial \mathcal{L}}{\partial k}\left(\frac{\partial \mathcal{M}}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial x} + \frac{\partial \mathcal{M}}{\partial x}\Big|_{\mathcal{L}\text{ fixed}}\right) - \frac{\partial \mathcal{L}}{\partial x}\frac{\partial \mathcal{M}}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial k} \\
&= \frac{\partial \mathcal{L}}{\partial k}\left(1 + \sum_{m=1}^{\infty}\frac{\partial v_m}{\partial x}\mathcal{L}^{-m-1}\right) \\
&= 1 + (\text{powers} < 0 \text{ of } k).
\end{aligned}
$$

While the right hand side

$$
\frac{\partial \widehat{\mathcal{L}}}{\partial k}\frac{\partial \widehat{\mathcal{M}}}{\partial x} - \frac{\partial \widehat{\mathcal{L}}}{\partial x}\frac{\partial \widehat{\mathcal{M}}}{\partial k} = (\text{powers} \geq 0 \text{ of } k).
$$

Comparing powers of both sides, we have

(5.4) $$\qquad\qquad \{\mathcal{L}, \mathcal{M}\} = \{\widehat{\mathcal{L}}, \widehat{\mathcal{M}}\} = 1.$$

Next, taking derivative of (5.2) with respect to t_n, we have

$$
\begin{pmatrix} \frac{\partial f}{\partial \mathcal{L}} & \frac{\partial f}{\partial \mathcal{M}} \\ \frac{\partial g}{\partial \mathcal{L}} & \frac{\partial g}{\partial \mathcal{M}} \end{pmatrix}
\begin{pmatrix} \frac{\partial \mathcal{L}}{\partial t_n} \\ \frac{\partial \mathcal{M}}{\partial t_n} \end{pmatrix}
= \begin{pmatrix} \frac{\partial \widehat{\mathcal{L}}}{\partial t_n} \\ \frac{\partial \widehat{\mathcal{M}}}{\partial t_n} \end{pmatrix}.
$$

Using (5.3) and (5.4), this is equivalent to

$$
\begin{pmatrix} \frac{\partial \mathcal{M}}{\partial x} & -\frac{\partial \mathcal{L}}{\partial x} \\ -\frac{\partial \mathcal{M}}{\partial k} & \frac{\partial \mathcal{L}}{\partial k} \end{pmatrix}
\begin{pmatrix} \frac{\partial \mathcal{L}}{\partial t_n} \\ \frac{\partial \mathcal{M}}{\partial t_n} \end{pmatrix}
= \begin{pmatrix} \frac{\partial \widehat{\mathcal{M}}}{\partial x} & -\frac{\partial \widehat{\mathcal{L}}}{\partial x} \\ -\frac{\partial \widehat{\mathcal{M}}}{\partial k} & \frac{\partial \widehat{\mathcal{L}}}{\partial k} \end{pmatrix}
\begin{pmatrix} \frac{\partial \widehat{\mathcal{L}}}{\partial t_n} \\ \frac{\partial \widehat{\mathcal{M}}}{\partial t_n} \end{pmatrix}.
$$

Hence we have

$$
\begin{aligned}
\frac{\partial \mathcal{M}}{\partial x}\frac{\partial \mathcal{L}}{\partial t_n} - \frac{\partial \mathcal{L}}{\partial x}\frac{\partial \mathcal{M}}{\partial t_n} &= \frac{\partial \widehat{\mathcal{M}}}{\partial x}\frac{\partial \widehat{\mathcal{L}}}{\partial t_n} - \frac{\partial \widehat{\mathcal{L}}}{\partial x}\frac{\partial \widehat{\mathcal{M}}}{\partial t_n} = (\text{powers} > 0 \text{ of } k) \\
-\frac{\partial \mathcal{M}}{\partial k}\frac{\partial \mathcal{L}}{\partial t_n} + \frac{\partial \mathcal{L}}{\partial k}\frac{\partial \mathcal{M}}{\partial t_n} &= -\frac{\partial \widehat{\mathcal{M}}}{\partial k}\frac{\partial \widehat{\mathcal{L}}}{\partial t_n} + \frac{\partial \widehat{\mathcal{L}}}{\partial k}\frac{\partial \widehat{\mathcal{M}}}{\partial t_n} = (\text{powers} \geq 0 \text{ of } k).
\end{aligned}
$$

We can rewrite the left hand sides of these equations as

$$
\begin{aligned}
&\left(\frac{\partial \mathcal{M}}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial x} + \frac{\partial \mathcal{M}}{\partial x}\Big|_{\mathcal{L}\text{ fixed}}\right)\frac{\partial \mathcal{L}}{\partial t_n} - \frac{\partial \mathcal{L}}{\partial x}\left(\frac{\partial \mathcal{M}}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial t_n} + \frac{\partial \mathcal{M}}{\partial t_n}\Big|_{\mathcal{L}\text{ fixed}}\right) \\
&= \left(1 + \sum_{m=1}^{\infty}\frac{\partial v_m}{\partial x}\mathcal{L}^{-m-1}\right)\frac{\partial \mathcal{L}}{\partial t_n} - \frac{\partial \mathcal{L}}{\partial x}\left(n\mathcal{L}^{n-1} + \sum_{m=1}^{\infty}\frac{\partial v_m}{\partial t_n}\mathcal{L}^{-m-1}\right) \\
&= -\frac{\partial (\mathcal{L}^n)_{>0}}{\partial x} + (\text{powers} \leq 0 \text{ of } k)
\end{aligned}
$$

and

$$(5.5) \qquad -\frac{\partial \mathcal{M}}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial k}\frac{\partial \mathcal{L}}{\partial t_n} + \frac{\partial \mathcal{L}}{\partial k}\left(\frac{\partial \mathcal{M}}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial t_n} + \frac{\partial \mathcal{M}}{\partial t_n}\Big|_{\mathcal{L}\text{ fixed}}\right)$$

$$=\frac{\partial \mathcal{L}}{\partial k}\left(n\mathcal{L}^{n-1} + \sum_{m=1}^{\infty}\frac{\partial v_m}{\partial t_n}\mathcal{L}^{-m-1}\right)$$

$$=\frac{\partial (\mathcal{L}^n)_{>0}}{\partial k} + (\text{powers} < 0 \text{ of } k).$$

Comparing powers with the right hand side, we have

$$\begin{pmatrix} \frac{\partial \mathcal{M}}{\partial x} & -\frac{\partial \mathcal{L}}{\partial x} \\ -\frac{\partial \mathcal{M}}{\partial k} & \frac{\partial \mathcal{L}}{\partial k} \end{pmatrix}\begin{pmatrix} \frac{\partial \mathcal{L}}{\partial t_n} \\ \frac{\partial \mathcal{M}}{\partial t_n} \end{pmatrix} = \begin{pmatrix} -\frac{\partial (\mathcal{L}^n)_{>0}}{\partial x} \\ \frac{\partial (\mathcal{L}^n)_{>0}}{\partial k} \end{pmatrix},$$

or

$$\begin{pmatrix} \frac{\partial \mathcal{L}}{\partial t_n} \\ \frac{\partial \mathcal{M}}{\partial t_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial k} & \frac{\partial \mathcal{L}}{\partial x} \\ \frac{\partial \mathcal{M}}{\partial k} & \frac{\partial \mathcal{M}}{\partial x} \end{pmatrix}\begin{pmatrix} -\frac{\partial (\mathcal{L}^n)_{>0}}{\partial x} \\ \frac{\partial (\mathcal{L}^n)_{>0}}{\partial k} \end{pmatrix},$$

i.e.

$$(5.6) \qquad \frac{\partial \mathcal{L}}{\partial t_n} = \{(\mathcal{L}^n)_{>0}, \mathcal{L}\} \qquad \text{and} \qquad \frac{\partial \mathcal{M}}{\partial t_n} = \{(\mathcal{L}^n)_{>0}, \mathcal{M}\}.$$

Since the negative powers part of (5.5) vanishes, rewriting the last equality in (5.5), we have

$$\frac{\partial (\mathcal{L}^n)_{>0}}{\partial k} = \frac{\partial}{\partial k}\left(\mathcal{L}^n - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial v_m}{\partial t_n}\mathcal{L}^{-m}\right).$$

Integrating with respect to k, we get

$$(5.7) \qquad (\mathcal{L}^n)_{>0} = \mathcal{L}^n - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial v_m}{\partial t_n}\mathcal{L}^{-m} - (\mathcal{L}^n)_0.$$

Taking derivative of (5.2) with respect to s, as the case of t_n, we have

$$\begin{pmatrix} \frac{\partial \mathcal{M}}{\partial x} & -\frac{\partial \mathcal{L}}{\partial x} \\ -\frac{\partial \mathcal{M}}{\partial k} & \frac{\partial \mathcal{L}}{\partial k} \end{pmatrix}\begin{pmatrix} \frac{\partial \mathcal{L}}{\partial s} \\ \frac{\partial \mathcal{M}}{\partial s} \end{pmatrix} = \begin{pmatrix} \frac{\partial \widehat{\mathcal{M}}}{\partial x} & -\frac{\partial \widehat{\mathcal{L}}}{\partial x} \\ -\frac{\partial \widehat{\mathcal{M}}}{\partial k} & \frac{\partial \widehat{\mathcal{L}}}{\partial k} \end{pmatrix}\begin{pmatrix} \frac{\partial \widehat{\mathcal{L}}}{\partial s} \\ \frac{\partial \widehat{\mathcal{M}}}{\partial s} \end{pmatrix}.$$

Now we have

$$\frac{\partial \mathcal{M}}{\partial x}\frac{\partial \mathcal{L}}{\partial s} - \frac{\partial \mathcal{L}}{\partial x}\frac{\partial \mathcal{M}}{\partial s} = \left(\frac{\partial \mathcal{M}}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial x} + \frac{\partial \mathcal{M}}{\partial x}\Big|_{\mathcal{L}\text{ fixed}}\right)\frac{\partial \mathcal{L}}{\partial s} - \frac{\partial \mathcal{L}}{\partial x}\left(\frac{\partial \mathcal{M}}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial s} + \frac{\partial \mathcal{M}}{\partial s}\Big|_{\mathcal{L}\text{ fixed}}\right)$$

$$= \left(1 + \sum_{m=1}^{\infty}\frac{\partial v_m}{\partial x}\mathcal{L}^{-m-1}\right)\frac{\partial \mathcal{L}}{\partial s} - \frac{\partial \mathcal{L}}{\partial x}\left(\frac{N}{\mathcal{L}} + \sum_{m=1}^{\infty}\frac{\partial v_m}{\partial s}\mathcal{L}^{-m-1}\right)$$

$$= \frac{\partial}{\partial s}\left(\mathcal{L} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial v_m}{\partial x}\mathcal{L}^{-m}\right) - \frac{\partial}{\partial x}\left(\log \mathcal{L}^N - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial v_m}{\partial s}\mathcal{L}^{-m}\right).$$

From (5.7), the term in the first bracket is

$$(\mathcal{L})_{\geq 0} = k + u_1.$$

We denote the term in the second bracket as

$$\log \mathcal{Q} = \log \mathcal{L}^N - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial v_m}{\partial s} \mathcal{L}^{-m}.$$

Hence

$$\frac{\partial \mathcal{M}}{\partial x} \frac{\partial \mathcal{L}}{\partial s} - \frac{\partial \mathcal{L}}{\partial x} \frac{\partial \mathcal{M}}{\partial s} = \frac{\partial u_1}{\partial s} - \frac{\partial \log \mathcal{Q}}{\partial x}.$$

On the other hand,

$$-\frac{\partial \mathcal{M}}{\partial k} \frac{\partial \mathcal{L}}{\partial s} + \frac{\partial \mathcal{L}}{\partial k} \frac{\partial \mathcal{M}}{\partial s} = -\frac{\partial \mathcal{M}}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial k} \frac{\partial \mathcal{L}}{\partial s} + \frac{\partial \mathcal{L}}{\partial k} \left(\frac{\partial \mathcal{M}}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial s} + \frac{N}{\mathcal{L}} + \sum_{m=1}^{\infty} \frac{\partial v_m}{\partial s} \mathcal{L}^{-m-1} \right)$$

$$= \frac{\partial \log \mathcal{Q}}{\partial k}.$$

Hence we have

(5.8)
$$\begin{pmatrix} \frac{\partial \widehat{\mathcal{M}}}{\partial x} & -\frac{\partial \widehat{\mathcal{L}}}{\partial x} \\ -\frac{\partial \widehat{\mathcal{M}}}{\partial k} & \frac{\partial \widehat{\mathcal{L}}}{\partial k} \end{pmatrix} \begin{pmatrix} \frac{\partial \widehat{\mathcal{L}}}{\partial s} \\ \frac{\partial \widehat{\mathcal{M}}}{\partial s} \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{M}}{\partial x} & -\frac{\partial \mathcal{L}}{\partial x} \\ -\frac{\partial \mathcal{M}}{\partial k} & \frac{\partial \mathcal{L}}{\partial k} \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial s} \\ \frac{\partial \mathcal{M}}{\partial s} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial s} - \frac{\partial \log \mathcal{Q}}{\partial x} \\ \frac{\partial \log \mathcal{Q}}{\partial k} \end{pmatrix}.$$

On the other hand, since $\{\widehat{\mathcal{L}}, \widehat{\mathcal{M}}\} = 1$, we have the identity

$$\begin{pmatrix} \frac{\partial \widehat{\mathcal{M}}}{\partial x} & -\frac{\partial \widehat{\mathcal{L}}}{\partial x} \\ -\frac{\partial \widehat{\mathcal{M}}}{\partial k} & \frac{\partial \widehat{\mathcal{L}}}{\partial k} \end{pmatrix} \begin{pmatrix} \{\log \mathcal{P}, \widehat{\mathcal{L}}\} \\ \{\log \mathcal{P}, \widehat{\mathcal{M}}\} \end{pmatrix} = \begin{pmatrix} -\frac{\partial \log \mathcal{P}}{\partial x} \\ \frac{\partial \log \mathcal{P}}{\partial k} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \frac{\partial \widehat{\mathcal{M}}}{\partial x} & -\frac{\partial \widehat{\mathcal{L}}}{\partial x} \\ -\frac{\partial \widehat{\mathcal{M}}}{\partial k} & \frac{\partial \widehat{\mathcal{L}}}{\partial k} \end{pmatrix} \begin{pmatrix} \frac{\partial \widehat{\mathcal{L}}}{\partial s} - \{\log \mathcal{P}, \widehat{\mathcal{L}}\} \\ \frac{\partial \widehat{\mathcal{M}}}{\partial s} - \{\log \mathcal{P}, \widehat{\mathcal{M}}\} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial s} - \frac{\partial}{\partial x} \log \frac{\mathcal{Q}}{\mathcal{P}} \\ \frac{\partial}{\partial k} \log \frac{\mathcal{Q}}{\mathcal{P}} \end{pmatrix}.$$

But by the conditions given on $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{M}}$, we have

$$\frac{\partial \widehat{\mathcal{M}}}{\partial x} \left(\frac{\partial \widehat{\mathcal{L}}}{\partial s} - \{\log \mathcal{P}, \widehat{\mathcal{L}}\} \right) - \frac{\partial \widehat{\mathcal{L}}}{\partial x} \left(\frac{\partial \widehat{\mathcal{M}}}{\partial s} - \{\log \mathcal{P}, \widehat{\mathcal{M}}\} \right) = (\text{ powers} \geq 1 \text{ of } k),$$

$$-\frac{\partial \widehat{\mathcal{M}}}{\partial k} \left(\frac{\partial \widehat{\mathcal{L}}}{\partial s} - \{\log \mathcal{P}, \widehat{\mathcal{L}}\} \right) + \frac{\partial \widehat{\mathcal{L}}}{\partial k} \left(\frac{\partial \widehat{\mathcal{M}}}{\partial s} - \{\log \mathcal{P}, \widehat{\mathcal{M}}\} \right) = (\text{ powers} \geq -1 \text{ of } k).$$

But by the normalization on Q and P, we have

$$\frac{\partial u_1}{\partial s} - \frac{\partial}{\partial x} \log \frac{\mathcal{Q}}{\mathcal{P}} = (\text{ powers} \leq 0 \text{ of } k),$$

$$\frac{\partial}{\partial k} \log \frac{\mathcal{Q}}{\mathcal{P}} = (\text{ powers} \leq -2 \text{ of } k).$$

Comparing powers, all the terms vanish, i.e.

$$\frac{\partial u_1}{\partial s} = 0, \qquad \frac{\partial}{\partial x} \log \frac{\mathcal{Q}}{\mathcal{P}} = 0, \qquad \frac{\partial}{\partial k} \log \frac{\mathcal{Q}}{\mathcal{P}} = 0.$$

Since $\log \frac{Q}{P}$ contains powers ≤ -1 of k, it has to vanish identically. In other words, we have $P = Q$ and

$$(5.9) \qquad \log P = \log \mathcal{L}^N - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial v_m}{\partial s} \mathcal{L}^{-m}.$$

From (5.8), we have

$$\begin{pmatrix} \frac{\partial \mathcal{M}}{\partial x} & -\frac{\partial \mathcal{L}}{\partial x} \\ -\frac{\partial \mathcal{M}}{\partial k} & \frac{\partial \mathcal{L}}{\partial k} \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial s} \\ \frac{\partial \mathcal{M}}{\partial s} \end{pmatrix} = \begin{pmatrix} -\frac{\partial \log P}{\partial x} \\ \frac{\partial \log P}{\partial k} \end{pmatrix}.$$

or

$$(5.10) \qquad \frac{\partial \mathcal{L}}{\partial s} = \{\log P, \mathcal{L}\}, \qquad \frac{\partial \mathcal{M}}{\partial s} = \{\log P, \mathcal{M}\}.$$

Finally, from (5.7) (5.9), (5.6) and (5.10), we get

$$\frac{\partial \log P}{\partial t_n} - \frac{\partial (\mathcal{L}^n)_{\geq 0}}{\partial s} + \{\log P, (\mathcal{L}^n)_{>0}\}$$

$$= \frac{\partial}{\partial t_n} \left(\log \mathcal{L}^N - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial v_m}{\partial s} \mathcal{L}^{-m} \right) - \frac{\partial}{\partial s} \left(\mathcal{L}^n - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial v_m}{\partial t_n} \mathcal{L}^{-m} \right)$$

$$+ \frac{\partial \log P}{\partial k} \frac{\partial (\mathcal{L}^n)_{>0}}{\partial x} - \frac{\partial \log P}{\partial x} \frac{\partial (\mathcal{L}^n)_{>0}}{\partial k}$$

$$= \frac{\partial \log P}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial t_n} - \frac{\partial (\mathcal{L}^n)_{>0}}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial s} + \frac{\partial \log P}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial k} \frac{\partial (\mathcal{L}^n)_{>0}}{\partial x} - \frac{\partial \log P}{\partial x} \frac{\partial (\mathcal{L}^n)_{>0}}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial k}$$

$$= \frac{\partial \log P}{\partial \mathcal{L}} \left(\frac{\partial \mathcal{L}}{\partial t_n} + \frac{\partial (\mathcal{L}^n)_{>0}}{\partial x} \frac{\partial \mathcal{L}}{\partial k} \right) - \frac{\partial (\mathcal{L}^n)_{>0}}{\partial \mathcal{L}} \left(\frac{\partial \mathcal{L}}{\partial s} + \frac{\partial \log P}{\partial x} \frac{\partial \mathcal{L}}{\partial k} \right)$$

$$= \frac{\partial \log P}{\partial \mathcal{L}} \frac{\partial (\mathcal{L}^n)_{>0}}{\partial k} \frac{\partial \mathcal{L}}{\partial x} - \frac{\partial (\mathcal{L}^n)_{>0}}{\partial \mathcal{L}} \frac{\partial \log P}{\partial k} \frac{\partial \mathcal{L}}{\partial x} = 0.$$

The fact that $\mathcal{M}$ is the corresponding Orlov function follows from the characterization of Orlov functions in Proposition 2.4.

$\square$

Conversely, we have

Proposition 5.2. *If $(\mathcal{L}, P, \mathcal{M})$ is a solution of dcmKP hierarchy, then there exists a pair of functions (f, g) such that $\{f, g\} = 1$ and satisfies (5.1) in Proposition 5.1*

Proof. We let

$$f(k, x) = e^{-\operatorname{ad} \varphi(s=0, t_n=0)} k, \qquad g(k, x) = e^{-\operatorname{ad} \varphi(s=0, t_n=0)} x.$$

Then obviously $\{f, g\} = 1$. The proof that f, g satisfies the first two conditions in (5.1) is standard (see Proposition 1.5.2 in [TT95]). Since $\mathcal{L}, \mathcal{M}$ satisfis (5.10), the other two conditions follows from the identities

$$\frac{\partial f(\mathcal{L}, \mathcal{M})}{\partial s} = \{\log P, f(\mathcal{L}, \mathcal{M})\}, \qquad \frac{\partial g(\mathcal{L}, \mathcal{M})}{\partial s} = \{\log P, g(\mathcal{L}, \mathcal{M})\}$$

and the first two conditions. $\square$

6. $w_{1+\infty}$ SYMMETRY

We consider the $w_{1+\infty}$ action on the space of solutions of the dcmKP hierarchy. Explicitly speaking, we define an infinitesimal deformation of (f, g) by a Hamiltonian vector field,

$$(f, g) \longrightarrow (f, g) \circ \exp(-\varepsilon \operatorname{ad} F),$$

and the associated deformation

$$(\mathcal{L}, \mathcal{P}, \mathcal{M}) \longrightarrow (\mathcal{L}(\varepsilon), \mathcal{P}(\varepsilon), \mathcal{M}(\varepsilon))$$

of the solution of dcmKP hierarchy. Here $\operatorname{ad} F$ is regarded as a Hamiltonian vector field

$$\operatorname{ad} F = \frac{\partial F}{\partial k} \frac{\partial}{\partial x} - \frac{\partial F}{\partial x} \frac{\partial}{\partial k},$$

and ε is an infinitesimal parameter. The infinitesimal symmetry is the first order coefficient $\delta_F(\cdot)$ in the ε-expansion:

$$\mathcal{L}(\varepsilon) = \mathcal{L} + \varepsilon \delta_F \mathcal{L} + O(\varepsilon^2), \qquad \mathcal{M}(\varepsilon) = \mathcal{M} + \varepsilon \delta_F \mathcal{M} + O(\varepsilon^2).$$

By definition, if G is a function of $\mathcal{L}$ and $\mathcal{M}$, then

$$\delta_F G(\mathcal{L}, \mathcal{M}) = \frac{\partial G}{\partial \mathcal{L}} \delta_F \mathcal{L} + \frac{\partial G}{\partial \mathcal{M}} \delta_F \mathcal{M},$$

while the independent variables are invariant : $\delta_F t = 0$.

The infinitesimal symmetries of the dcmKP hierarchy is given by the following propositions.

Proposition 6.1. *The infinitesimal symmetry of $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{P}$ are given by*

$$\delta_F \mathcal{L} = \{F(\mathcal{L}, \mathcal{M})_{\leq 0}, \mathcal{L}\}, \qquad \delta_F \mathcal{M} = \{F(\mathcal{L}, \mathcal{M})_{\leq 0}, \mathcal{M}\},$$

$$\delta_F \log \mathcal{P} = \frac{\partial F(\mathcal{L}, \mathcal{M})_{<0}}{\partial s} + \{F(\mathcal{L}, \mathcal{M})_{\leq 0}, \log \mathcal{P}\}.$$

Proof. By definition, the twistor data (f, g) is deformed as

$$(f_\varepsilon(k, x), g_\varepsilon(k, x)) = \left(e^{-\varepsilon \operatorname{ad} F} f(k, x), e^{-\varepsilon \operatorname{ad} F} g(k, x) \right)$$

$$= \left(f + \varepsilon \left(\frac{\partial f}{\partial k} \frac{\partial F}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial F}{\partial k} \right), g + \varepsilon \left(\frac{\partial g}{\partial k} \frac{\partial F}{\partial x} - \frac{\partial g}{\partial x} \frac{\partial F}{\partial k} \right) \right) + O(\varepsilon^2).$$

Hence, from

$$\widehat{\mathcal{L}}(\varepsilon) = f_\varepsilon(\mathcal{L}(\varepsilon), \mathcal{M}(\varepsilon)), \qquad \widehat{\mathcal{M}}(\varepsilon) = g_\varepsilon(\mathcal{L}(\varepsilon), \mathcal{M}(\varepsilon)),$$

we read off the coefficients of ε:

$$\begin{pmatrix} \delta_F \widehat{\mathcal{L}} \\ \delta_F \widehat{\mathcal{M}} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial \mathcal{L}} & \frac{\partial f}{\partial \mathcal{M}} \\ \frac{\partial g}{\partial \mathcal{L}} & \frac{\partial g}{\partial \mathcal{M}} \end{pmatrix} \begin{pmatrix} \delta_F \mathcal{L} + \frac{\partial F}{\partial \mathcal{M}} \\ \delta_F \mathcal{M} - \frac{\partial F}{\partial \mathcal{L}} \end{pmatrix}.$$

Now as in the proof of Proposition 5.1, we have

$$\begin{pmatrix} \frac{\partial \widehat{\mathcal{M}}}{\partial x} & -\frac{\partial \widehat{\mathcal{L}}}{\partial x} \\ -\frac{\partial \widehat{\mathcal{M}}}{\partial k} & \frac{\partial \widehat{\mathcal{L}}}{\partial k} \end{pmatrix} \begin{pmatrix} \delta_F \widehat{\mathcal{L}} \\ \delta_F \widehat{\mathcal{M}} \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{M}}{\partial x} & -\frac{\partial \mathcal{L}}{\partial x} \\ -\frac{\partial \mathcal{M}}{\partial k} & \frac{\partial \mathcal{L}}{\partial k} \end{pmatrix} \begin{pmatrix} \delta_F \mathcal{L} + \frac{\partial F}{\partial \mathcal{M}} \\ \delta_F \mathcal{M} - \frac{\partial F}{\partial \mathcal{L}} \end{pmatrix}.$$

Comparing powers of k, we have

$$\frac{\partial \mathcal{M}}{\partial x}\left(\delta_F \mathcal{L} + \frac{\partial F}{\partial \mathcal{M}}\right) - \frac{\partial \mathcal{L}}{\partial x}\left(\delta_F \mathcal{M} - \frac{\partial F}{\partial \mathcal{L}}\right) = (\text{powers} > 0 \text{ of } k),$$

$$-\frac{\partial \mathcal{M}}{\partial k}\left(\delta_F \mathcal{L} + \frac{\partial F}{\partial \mathcal{M}}\right) + \frac{\partial \mathcal{L}}{\partial k}\left(\delta_F \mathcal{M} - \frac{\partial F}{\partial \mathcal{L}}\right) = (\text{powers} \geq 0 \text{ of } k).$$

As in the proof of Proposition 5.1, these give

$$\frac{\partial \mathcal{M}}{\partial x}\delta_F \mathcal{L} - \frac{\partial \mathcal{L}}{\partial x}\delta_F \mathcal{M} = -\frac{\partial}{\partial x}F(\mathcal{L},\mathcal{M})_{\leq 0},$$

$$-\frac{\partial \mathcal{M}}{\partial k}\delta_F \mathcal{L} + \frac{\partial \mathcal{L}}{\partial k}\delta_F \mathcal{M} = \frac{\partial}{\partial k}F(\mathcal{L},\mathcal{M})_{<0} = \frac{\partial}{\partial k}F(\mathcal{L},\mathcal{M})_{\leq 0}.$$

Hence

$$\delta_F \mathcal{L} = \{F(\mathcal{L},\mathcal{M})_{\leq 0},\mathcal{L}\}, \qquad \delta_F \mathcal{M} = \{F(\mathcal{L},\mathcal{M})_{\leq 0},\mathcal{M}\}.$$

Now from

$$\frac{\partial \mathcal{L}(\varepsilon)}{\partial s} = \{\log \mathcal{P}(\varepsilon), \mathcal{L}(\varepsilon)\},$$

we have

$$\frac{\partial}{\partial s}\delta_F \mathcal{L} = \{\delta_F \log \mathcal{P}, \mathcal{L}\} + \{\log \mathcal{P}, \delta_F \mathcal{L}\}.$$

Using the results above, we have

$$\{\delta_F \log \mathcal{P}, \mathcal{L}\} = \{\frac{\partial}{\partial s}F(\mathcal{L},\mathcal{M})_{\leq 0}, \mathcal{L}\} + \{F(\mathcal{L},\mathcal{M})_{\leq 0}, \frac{\partial \mathcal{L}}{\partial s}\} - \{\log \mathcal{P}, \{F(\mathcal{L},\mathcal{M})_{\leq 0}, \mathcal{L}\}\}$$

$$= \{\frac{\partial}{\partial s}F(\mathcal{L},\mathcal{M})_{\leq 0}, \mathcal{L}\} + \{F(\mathcal{L},\mathcal{M})_{\leq 0}, \{\log \mathcal{P}, \mathcal{L}\}\} + \{\log \mathcal{P}, \{\mathcal{L}, F(\mathcal{L},\mathcal{M})_{\leq 0}\}\}$$

$$= \{\frac{\partial}{\partial s}F(\mathcal{L},\mathcal{M})_{\leq 0}, \mathcal{L}\} + \{\{F(\mathcal{L},\mathcal{M})_{\leq 0}, \log \mathcal{P}\}, \mathcal{L}\}.$$

In other words,

$$\{\delta_F \log \mathcal{P} - \frac{\partial}{\partial s}F(\mathcal{L},\mathcal{M})_{\leq 0} - \{F(\mathcal{L},\mathcal{M})_{\leq 0}, \log \mathcal{P}\}, \mathcal{L}\} = 0.$$

Similarly, if we use

$$\frac{\partial \mathcal{M}(\varepsilon)}{\partial s} = \{\log \mathcal{P}(\varepsilon), \mathcal{M}(\varepsilon)\},$$

we have

$$\{\delta_F \log \mathcal{P} - \frac{\partial}{\partial s}F(\mathcal{L},\mathcal{M})_{\leq 0} - \{F(\mathcal{L},\mathcal{M})_{\leq 0}, \log \mathcal{P}\}, \mathcal{M}\} = 0.$$

The next lemma implies that

$$\delta_F \log \mathcal{P} - \frac{\partial}{\partial s}F(\mathcal{L},\mathcal{M})_{\leq 0} - \{F(\mathcal{L},\mathcal{M})_{\leq 0}, \log \mathcal{P}\}$$

is independent of k and x. Comparing powers of k, we get

$$\delta_F \log \mathcal{P} = \frac{\partial}{\partial s}F(\mathcal{L},\mathcal{M})_{<0} + \{F(\mathcal{L},\mathcal{M})_{\leq 0}, \log \mathcal{P}\}.$$

$\square$

Lemma 6.2. *If $\mathcal{A}$ is such that $\{\mathcal{A}, \mathcal{L}\} = 0$ and $\{\mathcal{A}, \mathcal{M}\} = 0$, then $\mathcal{A}$ is independent of k and x.*

Proof. $\{\mathcal{A}, \mathcal{L}\} = 0$ and $\{\mathcal{A}, \mathcal{M}\} = 0$ are equivalent to

$$\begin{pmatrix} \frac{\partial \mathcal{L}}{\partial x} & -\frac{\partial \mathcal{L}}{\partial k} \\ \frac{\partial \mathcal{M}}{\partial x} & -\frac{\partial \mathcal{M}}{\partial k} \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{A}}{\partial k} \\ \frac{\partial \mathcal{A}}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $\{\mathcal{L}, \mathcal{M}\} = 1$, this implies that

$$\frac{\partial \mathcal{A}}{\partial k} = 0, \qquad \frac{\partial \mathcal{A}}{\partial x} = 0.$$

In other words, $\mathcal{A}$ is independent of k and x. $\qquad\square$

Proposition 6.3. *The infinitesimal symmetries of the v_n's are given by*

$$\delta_F v_n = -\operatorname{Res} F(\mathcal{L}, \mathcal{M}) d_k \mathcal{B}_n.$$

Proof. The proof follows the same line as Proposition 14 in [TT95], see the proof of Proposition 2.7. We have

$$\delta_F \mathcal{M}\Big|_{\mathcal{L} \text{ fixed}} = \sum_{n=1}^{\infty} \delta_F v_n \mathcal{L}^{-n-1}.$$

This gives

$$\delta_F v_n = \operatorname{Res} \mathcal{L}^n \left(\delta_F \mathcal{M} - \frac{\partial \mathcal{M}}{\partial \mathcal{L}} \delta_F \mathcal{L} \right) d_k \mathcal{L} = \operatorname{Res} \mathcal{L}^n d_k F(\mathcal{L}, \mathcal{M})_{\leq 0}$$
$$= \operatorname{Res}(\mathcal{L}^n)_{>0} d_k F(\mathcal{L}, \mathcal{M}) = -\operatorname{Res} F(\mathcal{L}, \mathcal{M}) d_k B_n.$$

The second equality follows the same as the proof in Proposition 2.7. $\qquad\square$

Proposition 6.4. *(1) The infinitesimal symmetry of the dressing function φ is determined (up to a function of s) by the relation*

$$\nabla_{\delta_F, \varphi} \varphi = F(\mathcal{L}, \mathcal{M})_{\leq 0},$$

or equivalently, by

$$\delta_F \varphi = \frac{\operatorname{ad} \varphi}{e^{\operatorname{ad} \varphi} - 1} F(\mathcal{L}, \mathcal{M})_{\leq 0} = \frac{T}{e^{\operatorname{ad} T} - 1}\Bigg|_{T = \operatorname{ad} \varphi} F(\mathcal{L}, \mathcal{M})_{\leq 0},$$

where $\frac{T}{e^{\operatorname{ad} T} - 1}$ is understood as a power series of T.
(2) The infinitesimal symmetry of the function ϕ defined in Proposition 2.1 is given (up to a function of s) by

$$\delta_F \phi = -F(\mathcal{L}, \mathcal{M})_0$$

Proof. First we proof (2). Let $\phi(\varepsilon)$ be the function given by Proposition 2.1 corresponding to $\mathcal{L}(\varepsilon)$. Since the function $\phi(\varepsilon)$ is defined up to functions of s, it is sufficient to show that

$$\frac{\partial \delta_F \phi}{\partial t_n} = -\frac{\partial}{\partial t_n} F(\mathcal{L}, \mathcal{M})_0 = -\left(\frac{\partial}{\partial t_n} F(\mathcal{L}, \mathcal{M}) \right)_0.$$

Now

$$\frac{\partial \delta_F \phi}{\partial t_n} = \delta_F \frac{\partial \phi}{\partial t_n} = \delta_F(\mathcal{L}^n)_0 = \left(\{F(\mathcal{L}, \mathcal{M})_{\leq 0}, \mathcal{L}^n\}\right)_0$$
$$= \left(\{F(\mathcal{L}, \mathcal{M})_{\leq 0}, (\mathcal{L}^n)_{>0}\}\right)_0 = \left(\{F(\mathcal{L}, \mathcal{M}), (\mathcal{L}^n)_{>0}\}\right)_0$$
$$= -\left(\frac{\partial}{\partial t_n} F(\mathcal{L}, \mathcal{M})\right)_0.$$

This prove (2).

Now let $\varphi(\varepsilon)$ be the dressing function of $\mathcal{L}(\varepsilon)$. Comparing the coefficient of ε in

$$\mathcal{L}(\varepsilon) = e^{\mathrm{ad}\,\varphi(\varepsilon)} k, \qquad \mathcal{M}(\varepsilon) = e^{\mathrm{ad}\,\varphi(\varepsilon)}\left(\sum_{n=1}^{\infty} n t_n k^{n-1} + x + \frac{Ns}{k}\right),$$

we have

$$\delta_F \mathcal{L} = \{\nabla_{\delta_F,\varphi}\varphi, \mathcal{L}\}, \qquad \delta_F \mathcal{M} = \{\nabla_{\delta_F,\varphi}\varphi, \mathcal{M}\}.$$

Compare with Proposition 6.1, we have

$$\{\nabla_{\delta_F,\varphi}\varphi - F(\mathcal{L}, \mathcal{M})_{\leq 0}, \mathcal{L}\} = 0, \qquad \{\nabla_{\delta_F,\varphi}\varphi - F(\mathcal{L}, \mathcal{M})_{\leq 0}, \mathcal{M}\} = 0.$$

By Lemma 6.2, this implies that

(6.1)
$$\nabla_{\delta_F,\varphi}\varphi - F(\mathcal{L}, \mathcal{M})_{\leq 0}$$

is a constant independent of x. To determine this constant, we have to find the coefficient of k^0 in $\nabla_{\delta_F,\varphi}\varphi$. Writing $\varphi = \sum_{n=0}^{\infty} \varphi_n k^{-n}$. Comparing the k^0 term in the identity

$$\nabla_{t_n,\varphi}\varphi = -(\mathcal{L}^n)_{\leq 0},$$

we have $\frac{\partial \varphi_0}{\partial t_n} = -(\mathcal{L}^n)_0$. From Proposition 2.1, this implies that up to a function of s, $\varphi_0 = -\phi$. Hence the k^0 term in (6.1) is

$$\delta_F \varphi_0 - (F(\mathcal{L}, \mathcal{M}))_0 = -\delta_F \phi - (F(\mathcal{L}, \mathcal{M}))_0.$$

By the second part of the proposition we prove above, this vanishes (up to a function of s). Hence we have the first part of our proposition. $\square$

Proposition 6.5. *The infinitesimal symmetry of the function Φ defined by (2.10) is given by*

$$\delta_F \Phi = -\operatorname{Res} F(\mathcal{L}, \mathcal{M}) d \log \mathcal{P}.$$

Proof. Again, since the function $\Phi(t)$ is defined up to a function of s, it is sufficient to show that

$$\frac{\partial \delta_F \Phi}{\partial t_n} = -\frac{\partial}{\partial t_n}\left(\operatorname{Res} F(\mathcal{L}, \mathcal{M}) d \log \mathcal{P}\right).$$

From (2.10) and Proposition 6.3,

$$\frac{\partial \delta_F \Phi}{\partial t_n} = \delta_F \frac{\partial \Phi}{\partial t_n} = \delta_F \frac{\partial v_n}{\partial s} = \frac{\partial \delta_F v_n}{\partial s} = -\frac{\partial}{\partial s} \operatorname{Res} F(\mathcal{L}, \mathcal{M}) d_k (\mathcal{L}^n)_{\geq 0}$$

$$= -\operatorname{Res}\{\log \mathcal{P}, F(\mathcal{L}, \mathcal{M})\} d_k \mathcal{B}_n + \operatorname{Res} \frac{\partial (\mathcal{L}^n)_{\geq 0}}{\partial s} d_k F(\mathcal{L}, \mathcal{M})$$

$$= -\operatorname{Res} \left(\frac{\partial \log \mathcal{P}}{\partial k} \frac{\partial F(\mathcal{L}, \mathcal{M})}{\partial x} - \frac{\partial \log \mathcal{P}}{\partial x} \frac{\partial F(\mathcal{L}, \mathcal{M})}{\partial k} \right) \frac{\partial \mathcal{B}_n}{\partial k} dk$$

$$\quad + \operatorname{Res} \frac{\partial \log \mathcal{P}}{\partial t_n} d_k F(\mathcal{L}, \mathcal{M}) + \operatorname{Res} \left(\frac{\partial \log \mathcal{P}}{\partial k} \frac{\partial \mathcal{B}_n}{\partial x} - \frac{\partial \log \mathcal{P}}{\partial x} \frac{\partial \mathcal{B}_n}{\partial k} \right) \frac{\partial F(\mathcal{L}, \mathcal{M})}{\partial k} dk$$

$$= -\operatorname{Res}\{\mathcal{B}_n, F(\mathcal{L}, \mathcal{M})\} d_k \log \mathcal{P} + \operatorname{Res} \frac{\partial \log \mathcal{P}}{\partial t_n} d_k F(\mathcal{L}, \mathcal{M})$$

$$= -\frac{\partial}{\partial t_n} \operatorname{Res} F(\mathcal{L}, \mathcal{M}) d_k \log \mathcal{P}.$$

$$\square$$

Remark 6.6. Observe that in the special case $\mathcal{P} = k$, $\phi = \Phi$. In fact in this case

$$-\operatorname{Res} F(\mathcal{L}, \mathcal{M}) d_k \log \mathcal{P} = -\operatorname{Res} F(\mathcal{L}, \mathcal{M}) d \log k = -F(\mathcal{L}, \mathcal{M})_0.$$

Remark 6.7. Compare the infinitesimal symmetries of $\mathcal{L}, \mathcal{P}, v_n, \Phi, \varphi$ given in the Propositions above with the t_n flows of this quantities, it suggests that the Hamiltonian vector field generate by the function $F(k, x) = -k^n$ is equivalent to $\frac{\partial}{\partial t_n}$. The discrepancy between the t_n flow of $\mathcal{M}$ with the infinitesimal symmetry of $\mathcal{M}$ when $F(k, x) = -k^n$ is because $\mathcal{M}$ depends explicitly on t_n, but we enforce $\delta_F t_n = 0$.

It is worth notice that if $\mathcal{L}$ is a solution to our dcmKP hierarchy with $\mathcal{P} = k$ and (f, g) an associated twistor data, then for any function $F(k, x)$, since

$$\frac{\partial F(\mathcal{L}, \mathcal{M})}{\partial s} = \{\log \mathcal{P}, F(\mathcal{L}, \mathcal{M})\} = \frac{1}{k} \frac{\partial F(\mathcal{L}, \mathcal{M})}{\partial x},$$

we have

$$\frac{\partial (F(\mathcal{L}, \mathcal{M}))_{<0}}{\partial s} = \frac{1}{k} \frac{\partial (F(\mathcal{L}, \mathcal{M}))_{\leq 0}}{\partial x}.$$

Hence from Proposition 6.1, we have

$$\delta_F \log \mathcal{P} = 0.$$

In other words, the class of special solutions $\mathcal{P} = k$ is stable under the $w_{1+\infty}$ action.

6.1. Symmetries extended to tau functions.

The above symmetries can be extended to tau functions as follows.

Proposition 6.8. *The infinitesimal symmetries of the tau function is given (up to a function of s) by*

$$\delta_F \log \tau = -\operatorname{Res} F^x(\mathcal{L}, \mathcal{M}) d_k \mathcal{L},$$

where $F^x(k, x)$ is a primitive function of $F(k, x)$ normalized as

$$F^x = \int_0^x F(k, y) dy.$$

It is compatible with the flows:

$$\frac{\partial}{\partial t} \delta_F \log \tau = \delta_F \frac{\partial \log \tau}{\partial t},$$

where $t = s$ or t_n's.

Proof. Let

$$F^x(\mathcal{L}, \mathcal{M}) = \sum_{m \in \mathbb{Z}} F^m(t) \mathcal{L}^m,$$

so that $\operatorname{Res} F^x(\mathcal{L}, \mathcal{M}) d_k \mathcal{L} = F_{-1}(t)$. Then we have to show that

$$(6.2) \qquad\qquad -\frac{\partial F_{-1}(t)}{\partial t} = \delta_F \frac{\partial \log \tau}{\partial t},$$

for $t = s$ or t_n's. Since the term $F_m(t)$'s come purely from $\mathcal{M}$, we have

$$\sum_{m \in \mathbb{Z}} \frac{\partial F_m(t)}{\partial t} \mathcal{L}^m = \frac{\partial F^x}{\partial \mathcal{M}} \frac{\partial \mathcal{M}}{\partial t}\Big|_{\mathcal{L} \text{ fixed}} = F(\mathcal{L}, \mathcal{M}) \frac{\partial \mathcal{M}}{\partial t}\Big|_{\mathcal{L} \text{ fixed}}$$

Hence as we have seen in the proof of Proposition 2.7,

$$\frac{\partial F_{-1}(t)}{\partial t} = \operatorname{Res} F(\mathcal{L}, \mathcal{M}) \frac{\partial \mathcal{M}}{\partial t}\Big|_{\mathcal{L} \text{ fixed}} d_k \mathcal{L} = \operatorname{Res} F(\mathcal{L}, \mathcal{M}) d_k \mathcal{A},$$

where $\mathcal{A} = \mathcal{B}_n$ if $t = t_n$ and $\mathcal{A} = \log \mathcal{P}$ if $t = s$. Since $\frac{\partial \log \tau}{\partial t} = v_n$ if $t = t_n$ and $\frac{\partial \log \tau}{\partial t} = \Phi$ if $t = s$, it follows from Propositions 6.3 and 6.5 that (6.2) holds. $\square$

The $w_{1+\infty}$ algebra structure is reflected as follows.

Proposition 6.9. *For the functions $F_1(k, x)$ and $F_2(k, x)$, the infinitesimal symmetries obey the commutation relations:*

$$[\delta_{F_1}, \delta_{F_2}] \log \tau = \delta_{\{F_1, F_2\}} \log \tau + c(F_1, F_2),$$
$$[\delta_{F_1}, \delta_{F_2}] \mathcal{K} = \delta_{\{F_1, F_2\}} \mathcal{K}, \qquad \mathcal{K} \in \{\mathcal{L}, \mathcal{M}, \mathcal{P}\},$$
$$[\delta_{F_1}, \delta_{F_2}] \varphi = \delta_{\{F_1, F_2\}} \varphi,$$
$$[\delta_{F_1}, \delta_{F_2}] \Phi = \delta_{\{F_1, F_2\}} \Phi,$$
$$[\delta_{F_1}, \delta_{F_2}] \phi = \delta_{\{F_1, F_2\}} \phi,$$

where

$$c(F_1, F_2) = \operatorname{Res} F_1(k, 0) d_k F_2(k, 0),$$

a cocycle of the $w_{1+\infty}$ algebra.

Proof. The proof of the first identity follows exactly the same as Proposition 16 in [TT95]. The second and fourth identities follows from the first one and the consistency of the infinitesimal symmetries of τ with time flows.

To prove the last identity, we have

$$
\begin{aligned}
\delta_{F_1}\delta_{F_2}\phi &= -\delta_{F_1}\Big(F_2(\mathcal{L},\mathcal{M})\Big)_0 \\
&= -\Big(\{(F_1(\mathcal{L},\mathcal{M}))_{\leq 0}, F_2(\mathcal{L},\mathcal{M})\}\Big)_0 \\
&= -\left(\frac{\partial(F_1(\mathcal{L},\mathcal{M}))_{\leq 0}}{\partial k}\frac{\partial F_2(\mathcal{L},\mathcal{M})}{\partial x} - \frac{\partial(F_1(\mathcal{L},\mathcal{M}))_{\leq 0}}{\partial x}\frac{\partial F_2(\mathcal{L},\mathcal{M})}{\partial k}\right)_0 \\
&= -\left(\frac{\partial(F_1(\mathcal{L},\mathcal{M}))_{\leq 0}}{\partial k}\frac{\partial(F_2(\mathcal{L},\mathcal{M}))_{> 0}}{\partial x} - \frac{\partial(F_1(\mathcal{L},\mathcal{M}))_{\leq 0}}{\partial x}\frac{\partial(F_2(\mathcal{L},\mathcal{M}))_{> 0}}{\partial k}\right)_0
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\delta_{F_1},\delta_{F_2}]\Phi &= \delta_{F_1}\delta_{F_2}\Phi - \delta_{F_2}\delta_{F_1}\Phi \\
&= -\left(\frac{\partial(F_1(\mathcal{L},\mathcal{M}))_{\leq 0}}{\partial k}\frac{\partial(F_2(\mathcal{L},\mathcal{M}))_{> 0}}{\partial x} - \frac{\partial(F_1(\mathcal{L},\mathcal{M}))_{\leq 0}}{\partial x}\frac{\partial(F_2(\mathcal{L},\mathcal{M}))_{> 0}}{\partial k}\right)_0 \\
&\quad + \left(\frac{\partial(F_2(\mathcal{L},\mathcal{M}))_{\leq 0}}{\partial k}\frac{\partial(F_1(\mathcal{L},\mathcal{M}))_{> 0}}{\partial x} - \frac{\partial(F_2(\mathcal{L},\mathcal{M}))_{\leq 0}}{\partial x}\frac{\partial(F_1(\mathcal{L},\mathcal{M}))_{> 0}}{\partial k}\right)_0 \\
&= -\left(\frac{\partial F_1(\mathcal{L},\mathcal{M})}{\partial k}\frac{\partial F_2(\mathcal{L},\mathcal{M})}{\partial x} - \frac{\partial F_1(\mathcal{L},\mathcal{M})}{\partial x}\frac{\partial F_2(\mathcal{L},\mathcal{M})}{\partial k}\right)_0 \\
&= -\Big(\{F_1,F_2\}(\mathcal{L},\mathcal{M})\Big)_0 = \delta_{\{F_1,F_2\}}\phi.
\end{aligned}
$$

For the third identity, observe that from the proof of Proposition 6.4, we have $\varphi_0 = -\phi$. From

$$
\begin{aligned}
e^{\operatorname{ad}\varphi}x &= x + \sum_{n=1}^{\infty} v_n \mathcal{L}^{-n-1} \\
&= x + \sum_{n=1}^{\infty}(v_n + (\text{polynomials in } \{u_1,\ldots,u_{n-1},v_1,\ldots,v_{n-1}\}))k^{-n-1},
\end{aligned}
$$

we can prove by induction that

$$
\begin{aligned}
& v_n + (\text{polynomials in } \{u_1,\ldots,u_{n-1},v_1,\ldots,v_{n-1}\}) \\
&= -n\varphi_n + (\text{differential polynomials of } \{\phi_0,\ldots,\phi_{n-1}\}),
\end{aligned}
$$

where the differential is taken with respect to x. Hence, solving recursively, we have for $n \geq 1$,

$$
\varphi_n = -\frac{v_n}{n} + (\text{differential polynomials of } \{u_1,\ldots,u_{n-1},v_1,\ldots,v_{n-1}\}).
$$

Hence the third identity follows from the second and fifth identities and the consistency between the infinitesimal symmetries and the t-flows. $\square$

7. Concluding remarks

We have defined a dcmKP hierarchy which incorporate both the ones defined by Kupershmidt, Chang and Tu [Kup90, CT00] and Takebe [Tak02]. Our motivation is to define a tau function for dmKP hierarchy in Kupershmidt, Chang and Tu's version, so that it plays the role of transition between the dToda hierarchy and dKP hierarchy. From our point of view, a good tau function should generate all the coefficients u_n's in the formal power series $\mathcal{L}$. Hence, we find that it is necessary to introduce an extra flow s. In our dcmKP hierarchy, the special case $\mathcal{P} = k$ has a tau function with the desired property, namely it generates the coefficients u_n's. For general $\mathcal{P}$, it does not have this property. However, our approach can be directly generalized to several extra flows $s_1, s_2, \ldots, s_M$ with M auxiliary polynomials $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_M$ to govern the flows. If one of the P_i is equal to k, then we will find a good tau function. As a matter of fact, $\mathcal{P}$ need not be a polynomial. What we require is that the coefficient of the leading term is one.

Our dcmKP hierarchy can also be considered as a quasiclassical limit of a corresponding coupled modified KP (cmKP) hierarchy. It will be interesting to establish the existence of a tau function in the cmKP hierarchy, so that its quasiclassical limit is our dispersionless tau function.

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